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Mathematical Tripos Part II: Michaelmas Term 2024

Numerical Analysis – Lecture 8

Problem 2.18 (The advection equation) We look at the *advection equation* which we already considered in Lecture 6.

$$u_t = u_x, \qquad t \geq 0, \tag{2.6}$$

where $u = u(x,t)$. It is given with the initial condition $u(x,0) = \varphi(x)$. The exact solution of (2.6) is simply $u(x,t) = \varphi(x+t)$, a unilateral shift leftwards. This, however, does not mean that its numerical modelling is easy.

Example 2.19 (Downwind instability) 1) *Downwind instability*: Consider the discretization $\frac{\partial u_m(t)}{\partial x} \approx \frac{1}{2h}\left[u_m(t) - u_{m-1}(t)\right]$, so coming to the ODE $u_m'(t) = \frac{1}{2h}\left[u_m(t) - u_{m-1}(t)\right]$. For the Euler method, the outcome is

$$u_m^{n+1} = u_m^n + \mu(u_m^n - u_{m-1}^n), \quad n \in \mathbb{Z}_+.$$

We can analyze the stability of this method using Fourier analysis. The amplification factor is

$$H(\theta) = 1 + \mu - \mu e^{-\mathrm{i}\theta}.$$

We see that for $\theta = \pi/2$, $|H(\theta)|^2 = (1+\mu)^2 + \mu^2 > 1$, and so the method is unstable for all $\mu > 0$.

Method 2.20 (Upwind method) *Upwind scheme*: If we semidiscretize $\frac{\partial u_m(t)}{\partial x} \approx \frac{1}{h}\left[u_{m+1}(t) - u_m(t)\right]$, and solve the ODE again by Euler's method, then the result is

$$u_m^{n+1} = u_m^n + \mu(u_{m+1}^n - u_m^n), \quad n \in \mathbb{Z}_+ \tag{2.7}$$

The local error is $\mathcal{O}(k^2 + kh)$ which is $\mathcal{O}(h^2)$ for a fixed μ, hence convergence if the method is stable. We can again use Fourier analysis to analyze stability. The amplification factor is

$$H(\theta) = 1 - \mu + \mu e^{\mathrm{i}\theta}$$

and we see that $|H(\theta)| = |1 - \mu + \mu e^{\mathrm{i}\theta}| \leq |1-\mu| + \mu = 1$ for $\mu \in [0,1]$. Hence we have stability for $\mu \leq 1$. If $\mu > 1$, then note that $|H(\pi)| = |1 - 2\mu| > 1$, and so we have instability for $\mu > 1$.

Matlab demo: Download the Matlab GUI for *Solving the Advection Equation, Upwinding and Stability* from `https://www.damtp.cam.ac.uk/user/hf323/M21-II-NA/demos/index.html` and solve the advection equation (2.6) with the different methods provided in the demonstration. Experience what can go wrong when "winding" in the wrong direction!

What about the case when $0 \leq x \leq 1$ (bounded domain)? Recall from Lecture 6 when we considered the Euler method for the advection equation

$$u_m^{n+1} - u_m^n = \mu(u_{m+1}^n - u_m^n), \qquad m = 1 \ldots M\,.$$

We have $\boldsymbol{u}^{n+1} = A\boldsymbol{u}^n$, where

$$A = \begin{bmatrix} 1-\mu & \mu & & \\ & 1-\mu & \ddots & \\ & & \ddots & \mu \\ & & & 1-\mu \end{bmatrix},$$

but A is *not* normal, and although its eigenvalues are bounded by 1 for $\mu \leq 2$ (note $1-\mu$ is the only eigenvalue of A), it is the matrix induced norm of A that matters. For this example, it is easier to work with $\|A\|_{\infty\to\infty}$ which we see is given by $|1-\mu| + \mu$ (by the formula in Lecture 5), and this is smaller than 1 precisely when $\mu \leq 1$.

Method 2.21 (The leapfrog method) *Leap-frog method*: We semidicretize (2.6) as $\frac{\partial u_m(t)}{\partial x} \approx \frac{1}{2h}\left[u_{m+1}(t) - u_{m-1}(t)\right]$, but now solve the ODE with the second-order *midpoint rule*

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_{n-1} + 2k\boldsymbol{f}(t_n, \boldsymbol{y}_n), \qquad n \in \mathbb{Z}_+\,.$$

The outcome is the two-step *leapfrog* method

$$u_m^{n+1} = \mu\left(u_{m+1}^n - u_{m-1}^n\right) + u_m^{n-1}. \tag{2.8}$$

The local error is now $\mathcal{O}(k^3 + kh^2) = \mathcal{O}(h^3)$.

We analyse stability by the Fourier technique, assuming that we are solving a Cauchy problem. Thus, proceeding as before,

$$\widehat{u}^{n+1}(\theta) = \mu\left(e^{i\theta} - e^{-i\theta}\right)\widehat{u}^n(\theta) + \widehat{u}^{n-1}(\theta) \tag{2.9}$$

whence

$$\widehat{u}^{n+1}(\theta) - 2i\mu\,\sin\theta\,\widehat{u}^n(\theta) - \widehat{u}^{n-1}(\theta) = 0, \qquad n \in \mathbb{Z}_+\,,$$

and our goal is to determine values of μ such that $|\widehat{u}^n(\theta)|$ is uniformly bounded for all n, θ.

This is a difference equation $w_{n+1} + bw_n + cw_{n-1} = 0$ with the general solution $w_n = c_1\lambda_1^n + c_2\lambda_2^n$, where λ_1, λ_2 are the roots of the characteristic equation $\lambda^2 + b\lambda + c = 0$, and c_1, c_2 are constants, dependent on the initial values w_0 and w_1. If $\lambda_1 = \lambda_2$, then solution is $w_n = (c_1 + c_2 n)\lambda^n$. In our case, we obtain

$$\lambda_{1,2}(\theta) = i\mu\sin\theta \pm \sqrt{1 - \mu^2\sin^2\theta}\,.$$

Stability is equivalent to $|\lambda_{1,2}(\theta)| \le 1$ for all θ and this is true if and only if $\mu \le 1$.

Problem 2.22 (The wave equation) Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad t \ge 0,$$

given with initial conditions $u(x, 0)$ and $u_t(x, 0) = \frac{\partial u}{\partial t}(x, 0)$. The usual approximation looks as follows

$$u_m^{n+1} - 2u_m^n + u_m^{n-1} = \mu(u_{m+1}^n - 2u_m^n + u_{m-1}^n)\,,$$

with the Courant number being now $\mu = k^2/h^2$.

The Fourier analysis (for Cauchy problem) provides

$$\widehat{u}^{n+1}(\theta) - 2\widehat{u}^n(\theta) + \widehat{u}^{n-1}(\theta) = -4\mu\sin^2\tfrac{\theta}{2}\,\widehat{u}^n(\theta)\,,$$

with the characteristic equation $\lambda^2 - 2(1 - 2\mu\sin^2\frac{\theta}{2})\lambda + 1 = 0$. The product of the roots is one, therefore stability (that requires the moduli of both λ to be at most one) is equivalent to the roots being complex conjugate, so we require

$$(1 - 2\mu\sin^2\tfrac{\theta}{2})^2 \le 1.$$

This condition is achieved if and only if $\mu = k^2/h^2 \le 1$.