Modular structure of Type IIB low energy expansion

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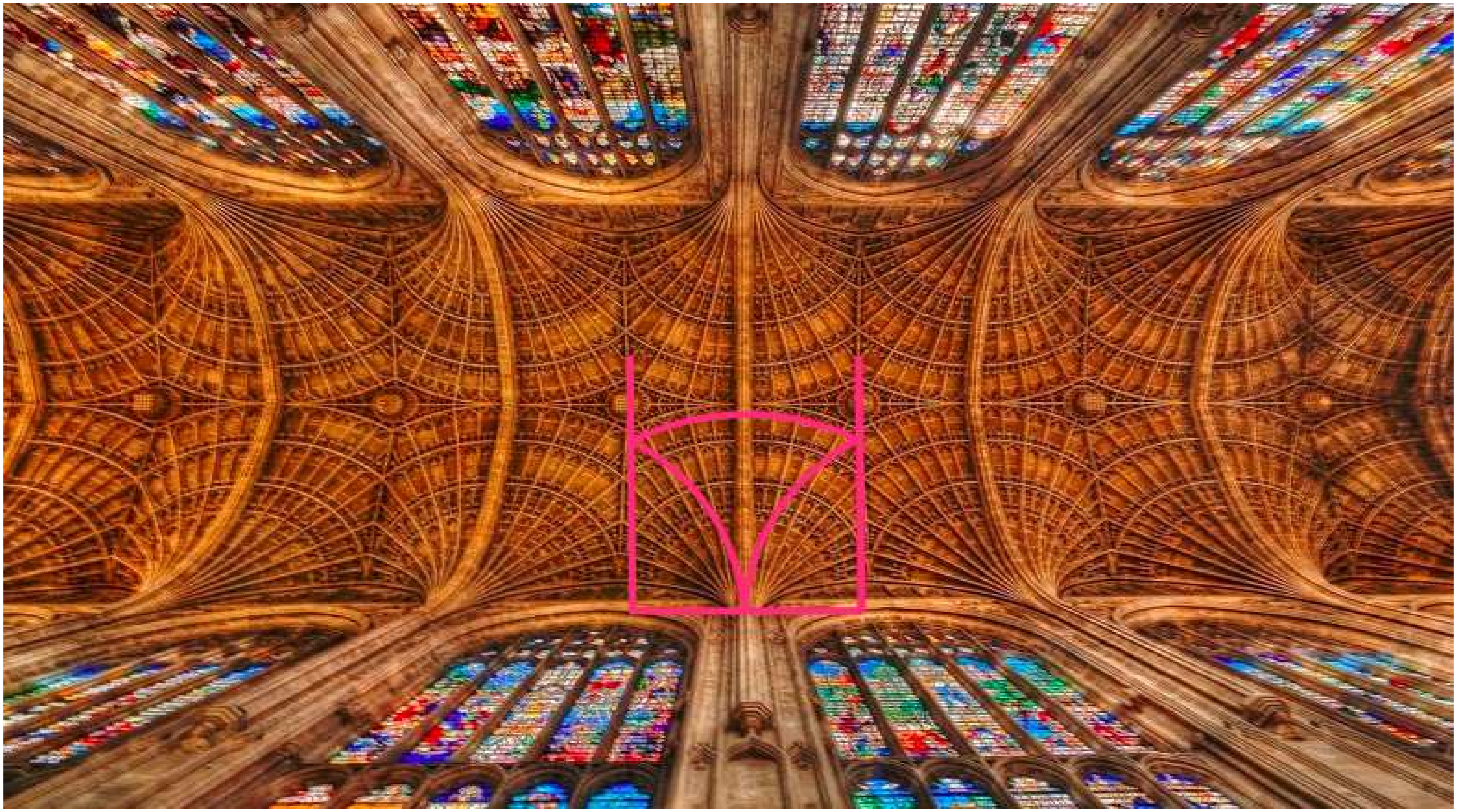
Bibliography

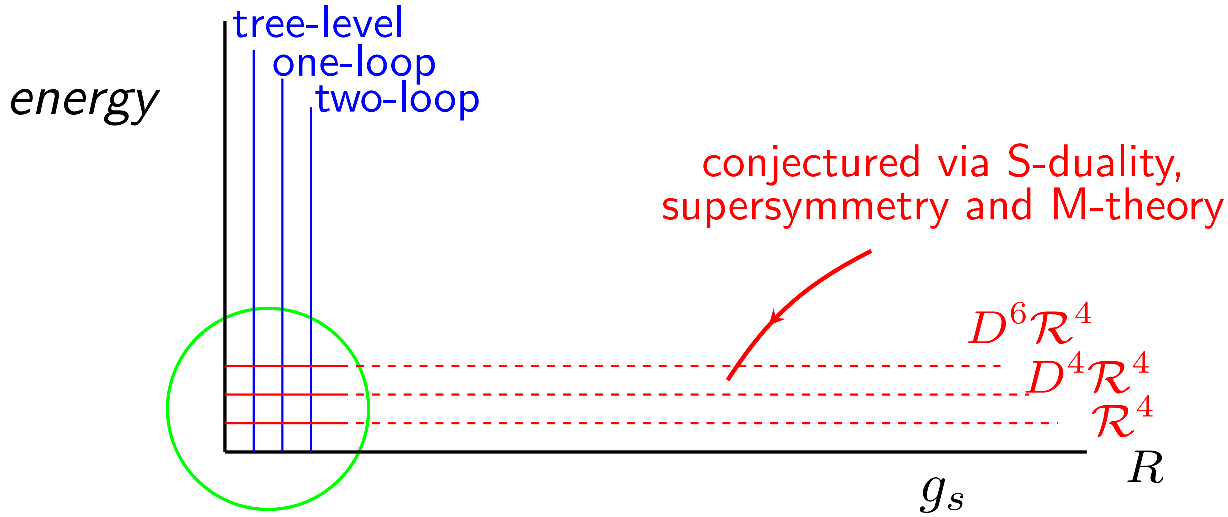
Based on

- ED, Michael Green, Pierre Vanhove, arXiv:1502.06698,
  On the modular structure of the genus-one Type II superstring low energy expansion
- ED, Michael Green, Boris Pioline, Rudolfo Russo, arXiv:1405.6226; JHEP 1501 (2015) 031,
  Matching the $D^6\mathcal{R}^4$ interaction at two-loops
- ED, Michael Green, arXiv:1308.4597; Journal of Number Theory, Vol 144 (2014) 111-150,
  Zhang-Kawazumi invariants and Superstring Amplitudes

... Modular Structure ...



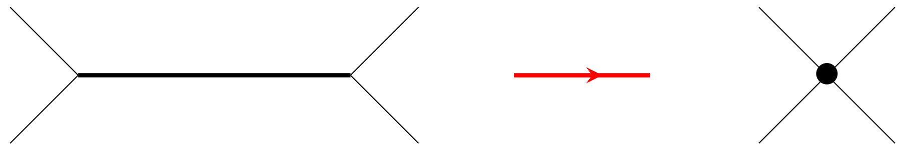
Expansions of Type IIB Superstring Theory



- Superstring Perturbation Theory in powers of g_s
  - holds for all energies
  - but for weak coupling g_s only
- Classical supergravity (R)
  - leading low energy expansion of string theory
  - holds for all couplings g_s
- String induced effective interactions $\mathcal{R}^4, D^4\mathcal{R}^4, D^6\mathcal{R}^4$
  - Evaluated in perturbation theory for $g_s \ll 1$
  - Conjectured for all couplings via S-duality, supersmmetry and M-theory

Effective Interactions

Exchange of massive string states produces local effective interactions.



- Four-graviton amplitude in Type II at tree-level,

$$\mathcal{A}_0 = \kappa^2 \mathcal{R}^4 \frac{1}{stu} \frac{\Gamma(1-s)\,\Gamma(1-t)\,\Gamma(1-u)}{\Gamma(1+s)\,\Gamma(1+t)\,\Gamma(1+u)}$$

  - $\kappa^2 =$ Newton's constant in 10 dimensions;
  - $\mathcal{R}^4 =$ unique maximally supersymmetric contraction of 4 Weyl tensors
  - $s_{ij} = -\alpha' k_i \cdot k_j/2$, $s = s_{12}$, $t = s_{13}$, $u = s_{14}$ with $s+t+u=0$

- Low energy expansion corresponds to $|s|, |t|, |u| \ll 1$

$$\frac{1}{stu} + 2\zeta(3) + \zeta(5)(s^2+t^2+u^2) + 2\zeta(3)^2 stu + \cdots$$

massless $\qquad \mathcal{R}^4 \qquad D^4\mathcal{R}^4 \qquad D^6\mathcal{R}^4$

D-instantons and Eisenstein series

Cambridge 1997 $\cdots$ [Green Gutperle]

- Conjectured full $\mathcal{R}^4$ effective interaction from D-instanton calculation,

$$(T_2)^{\frac{1}{2}}\, E_{\frac{3}{2}}(T)\, \mathcal{R}^4 \qquad\qquad T = T_1 + iT_2, \qquad T_2 = \frac{1}{g_s}$$

- The (non-holomorphic) Eisenstein series,

$$E_s(T) = \sum_{(m,n)\neq(0,0)} \frac{(T_2)^s}{\pi^s |mT+n|^{2s}}$$

  - Modular invariant under S-duality group $SL(2,\mathbb{Z})$ of Type IIB;
  - satisfies a Laplace-eigenvalue equation,

$$\Delta E_s = s(s-1)E_s \qquad\qquad \Delta = 4\, T_2^2\, \partial_T \partial_{\bar{T}}$$

  - and admits the following asymptotics near the cusp $T_2 \to \infty$,

$$E_s(T,\bar{T}) = \frac{2\zeta(2s)}{\pi^s} T_2^s + \frac{2\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\pi^{s-\frac{1}{2}}} T_2^{1-s} + \mathcal{O}(e^{-2\pi T_2})$$

  - Perturbative contributions only from genus 0 and 1.

Supersymmetry and S-duality

- Laplace-eigenvalue eq results from space-time supersymmetry [Green, Sethi, 1998]
  - Eisenstein series = unique modular solution with polynomial growth at cusp

- Predicts vanishing contributions for high enough loop order,

$\mathcal{R}^4$	1/2 BPS	$h \geq 2$	$E_{\frac{3}{2}}$
$D^4\mathcal{R}^4$	1/4 BPS	$h \geq 3$	$E_{\frac{5}{2}}$
$D^6\mathcal{R}^4$	1/8 BPS	$h \geq 4$	$(\Delta - 12)\mathcal{E}_{D^6\mathcal{R}^4} = (E_{\frac{3}{2}})^2$

[Green, Gutperle, Vanhove 1997; Green, Vanhove 2005]

- Predicts relations between non-vanishing contributions (such as with tree-level),

$\mathcal{R}^4$	$h = 1$	[Green, Gutperle 1997]
$D^4\mathcal{R}^4$	$h = 2$	[ED, Gutperle, Phong 2005]
$D^6\mathcal{R}^4$	$h = 2$	[ED, Green, Pioline, Russo 2014]
	$h = 3$	[Gomez, Mafra 2013]

Focus of this talk

- $D^6\mathcal{R}^4$ at two-loops.
  - involves a new modular object, the "Zhang-Kawazumi-invariant".

- Structure of $D^{2w}\mathcal{R}^4$ effective interactions for $w \geq 4$.
  - no longer governed by BPS;
  - at one loop produces rich structure of non-holomorphic modular forms.
  - natural generalization to two-loops (beyond the scope of this talk)

- In both cases, we will find that the *integrands* on moduli space
  - ⋆ of compact Riemann surfaces (without punctures),
  - ⋆ having integrated over all vertex operator positions,
    - obey families of interesting differential and algebraic equations;
    - specify $D^{2w}\mathcal{R}^4$ for un-compactified or compactified space-times.

$D^6\mathcal{R}^4$ at genus-two

- Start with Type II four-graviton amplitude at genus 2, [ED, Phong 2005]

$$\mathcal{A}_2 = \frac{\pi}{64}\kappa^2\mathcal{R}^4 \int_{\mathcal{M}_2} d\mu_2\, \mathcal{B}_2(s,t,u|\Omega)$$

$$\mathcal{B}_2 = \int_{\Sigma^4} \mathcal{Y} \wedge \bar{\mathcal{Y}} \exp \sum_{i<j} s_{ij} G(i,j)$$

  - $\mathcal{M}_2$ is the moduli space with Siegel volume form $d\mu_2$;
  - $G(i,j)$ is the scalar Green function;
  - $\mathcal{Y} = (s-t)\Delta(1,3) \wedge \Delta(4,2) + 2$ permutations;
  - $\Delta(i,j)$ is a holomorphic $(1,0)_i \otimes (1,0)_j$ form independent of s,t,u.

- Contributions produced to local effective interactions
  - $\mathcal{R}^4$: zero, since $\mathcal{Y}$ vanishes for $s=t=u=0$;
  - $D^4\mathcal{R}^4$: non-zero, $\mathcal{B}_2$ constant on $\mathcal{M}_2$;
  - $D^6\mathcal{R}^4$: non-zero, one power of G brought down in integral over Σ^4;

$$\mathcal{B}_2 = 32(s^2+t^2+u^2) + 192\, stu\, \varphi(\Omega) + \mathcal{O}(s^4,\cdots,u^4)$$

  - $\varphi(\Omega)$ coincides with the Zhang Kawazumi invariant [ED, Green 2013].

The Zhang-Kawazumi invariant for genus-two

- Definition of the ZK-invariant
  - Let A_I, B_I be canonical homology basis
  - ω_I holó (1,0) forms with $\oint_{A_I} \omega_J = \delta_{IJ}$ and $\oint_{B_I} \omega_J = \Omega_{IJ} = X_{IJ} + iY_{IJ}$;

$$8\varphi(\Omega) = \sum_{I,J,K,L} \left(Y^{-1}_{IJ} Y^{-1}_{KL} - 2Y^{-1}_{IL} Y^{-1}_{JK}\right) \int_{\Sigma^2} G(x,y) \omega_I(x) \overline{\omega_J(x)} \omega_K(y) \overline{\omega_L(y)}$$

  - equivalent to definition via Arakelov geometry [Zhang 2007, Kawazumi 2008]
  - related to the genus-two Faltings invariant [De Jong 2010]
  - invariant under the modular group $Sp(4,\mathbb{Z})$

- Direct evaluation of $\int_{\mathcal{M}_2} d\mu_2 \, \varphi(\Omega)$ appeared out of reach ... until ...

ZK satisfies a Laplace eigenvalue equation

- Theorem : genus-two ZK invariant satisfies remarkably simple equation,

$$(\Delta - 5)\varphi = -2\pi\delta_{SN}$$

  - Δ is the Laplace-Beltrami operator on $\mathcal{M}_2$ with Siegel metric;
  - δ_{SN} has support on separating node (into two genus-one surfaces)

[ED, Green, Pioline, R. Russo 2014]

- Using Theorem, the integral over $\mathcal{M}_2$ reduces to an integral over $\partial\mathcal{M}_2$,

$$\int_{\mathcal{M}_2} d\mu_2\, \varphi = \frac{1}{5}\int_{\mathcal{M}_2} d\mu_2 \big(\Delta\varphi + 2\pi\delta_{SN}\big) = \frac{2\pi^3}{45}$$

  - agrees with prediction from Supersymmetry, S-duality, M-theory on $\mathbb{T}^2$

Evidence and Proof

- Initial indications from $D^6\mathcal{R}^4$ interaction for compactification on $\mathbb{T}^d$,

$$\mathcal{E}^{(2)}_{D^6\mathcal{R}^4} = \int_{\mathcal{M}_2} d\mu_2 \, \varphi(\Omega) \, \Gamma_d(T_d|\Omega)$$

  - Γ_d is the torus partition function dependent on scalar vevs T_d; $T_0 \in SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2), \cdots, T_7 \in E_{8(8)}(\mathbb{Z})\backslash E_{8(8)}(\mathbb{R})/SU(8)$
  - Γ_d satisfies $(2\Delta - \Delta_{T_d} + d(3-d))\Gamma_d = 0$;
  - Supersymmetry & duality conjectured relation with genus-one $\mathcal{E}^{(1)}_{\mathcal{R}^4}$

$$\left(\Delta_{T_d} - \frac{6(4-d)(d+4)}{8-d}\right) \mathcal{E}^{(2)}_{D^6\mathcal{R}^4} = -\big(\mathcal{E}^{(1)}_{\mathcal{R}^4}\big)^2 + 40\zeta(3)\delta_{d,4}$$

- Further supported by asymptotic behavior of φ [De Jong 2012, Wentworth 1991]
- Direct proof using deformations of complex structures on Riemann surfaces
  - Δ on $\mathcal{M}_2$ obtained from insertions of 2 stress tensors, T_{zz} and $T_{\bar z \bar z}$

[ED, Green, Pioline, R. Russo 2014]

Generalizations of KZ-invariant

- The KZ-invariant exists for all genera $h \geq 2$ [Zhang 2007, Kawazumi 2008]
  - but does not satisfy a simple Laplace-eigenvalue eq for $h \geq 3$;
  - likely is not the correct object for string theory at $h \geq 3$.
- But the *integrands* on $\mathcal{M}_2$ for the coefficients of $D^8\mathcal{R}^4$, $D^{10}\mathcal{R}^4, \cdots$
  - do naturally emerge from string theory;
  - are modular invariants which generalize ZK;
  - satisfy more complicated Laplace-type equations
    [ED, Green, Vanhove] ... in progress ...
- Actually, even the corresponding genus-one problem remains to be explored $\cdots$

Genus-one effective interactions

- The genus-one four-graviton amplitude is an integral over moduli space $\mathcal{M}_1$,

$$\mathcal{A}_1^{(4)} = 2\pi\kappa^2\,\mathcal{R}^4 \int_{\mathcal{M}_1} d\mu_1\,\mathcal{B}_1(s,t,u|\tau)$$

- The partial amplitude $\mathcal{B}_1$ reduces to an integral over four copies of the torus Σ,

$$\mathcal{B}_1(s,t,u|\tau) = \left(\prod_{i=1}^4 \int_\Sigma \frac{d^2 z_i}{\tau_2} \right) \exp\Big\{ \sum_{1 \leq i < j \leq 4} s_{ij}\, G(z_i - z_j|\tau) \Big\}$$

  – The scalar Green function on Σ is a Fourier sum of torus momenta $(m,n) \in \mathbb{Z}^2$, where $z = \alpha + \beta\tau$ with $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$,

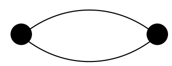
$$G(z|\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2}{\pi|m\tau+n|^2}\, e^{2\pi i(m\alpha - n\beta)}$$

Worldsheet Feynman diagrams

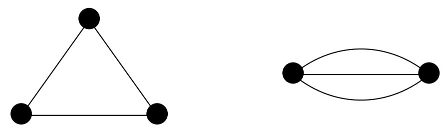
- The expansion in powers of s_{ij} may be organized in Feynman diagrams;
  - Each integration point z_i on Σ is represented by a vertex;
  - Each Green function $G(z_i - z_j|\tau)$ by a line —— between z_i and z_j;
  - diagrams with a single G ending in a point vanish by $\int_\Sigma d^2z G(z|\tau) = 0$
  - a diagram with w lines of G,
    - ⋆ has *weight* w;
    - ⋆ contributes to $D^{2w}\mathcal{R}^4$.

Worldsheet Feynman diagrams (connected)

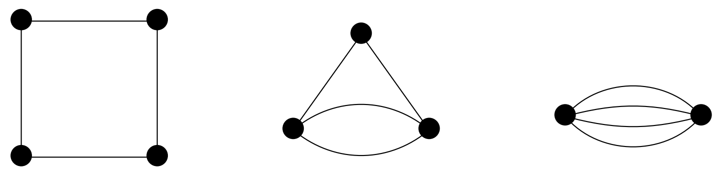
$D^4\mathcal{R}^4$



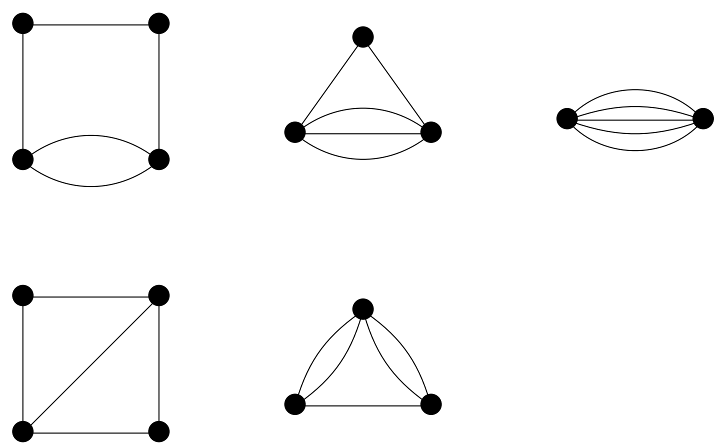
$D^6\mathcal{R}^4$



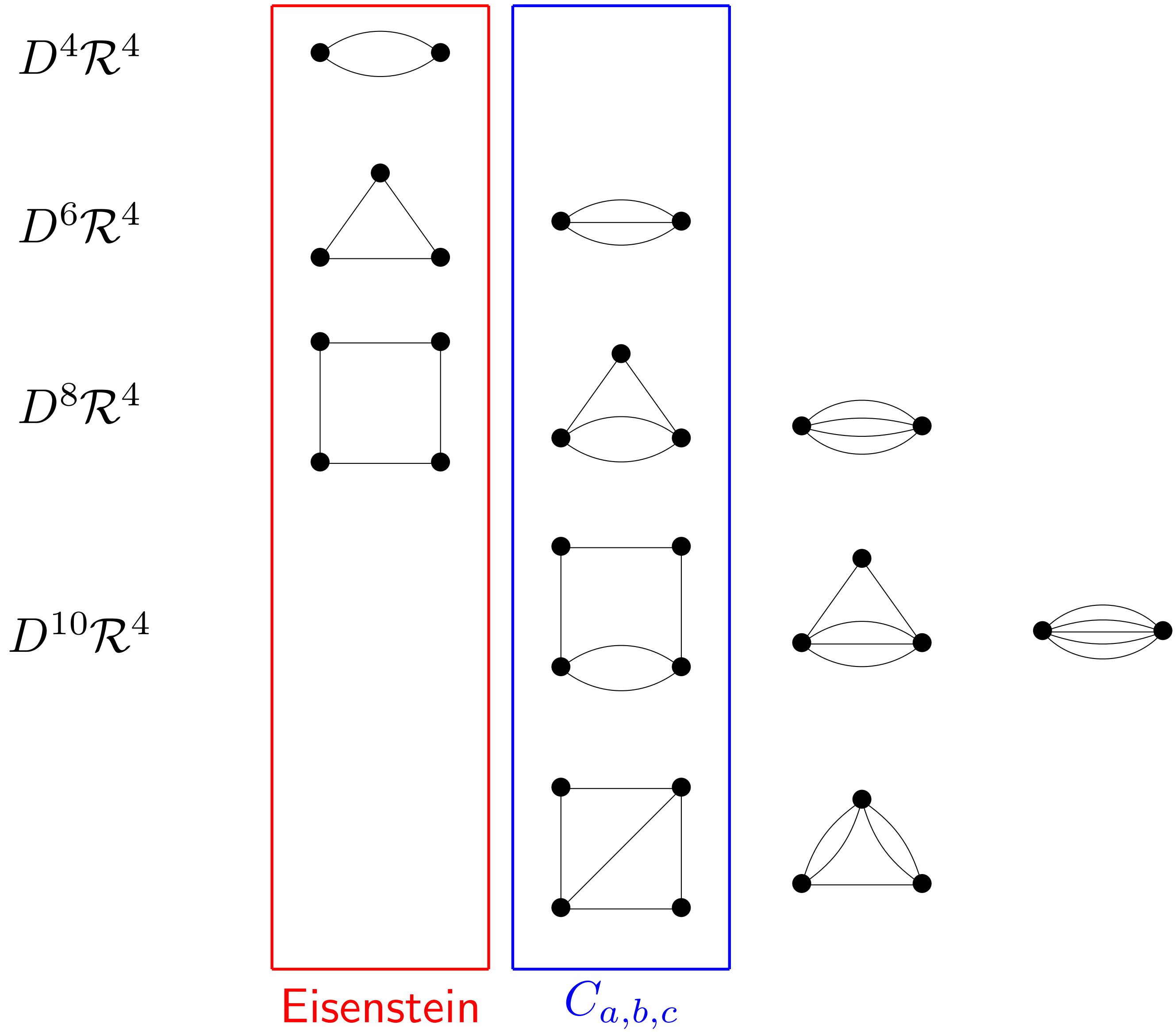
$D^8\mathcal{R}^4$



$D^{10}\mathcal{R}^4$



Worldsheet Feynman diagrams (connected)



Kronecker-Eisenstein series

- One-loop worldsheet Feynman diagrams generate Eisenstein series.
  - for example to order $s^2 + t^2 + u^2$

$$\int_\Sigma \frac{d^2 z}{\tau_2} G(z|\tau)^2 = \sum_{(m,n) \neq (0,0)} \frac{\tau_2^2}{\pi^2 |m\tau + n|^4} = E_2(\tau)$$

- Two-loop worldsheet Feynman diagrams generate "Kronecker-Eisenstein series".

$$C_{a_1,a_2,a_3}(\tau) = \sum_{(m_r,n_r) \neq (0,0)} \delta_{m,0}\, \delta_{n,0} \prod_{r=1}^{3} \left(\frac{\tau_2}{\pi |m_r \tau + n_r|^2} \right)^{a_r}$$

- The total worldsheet momenta $m = m_1 + m_2 + m_3$, $n = n_1 + n_2 + n_3$ vanish;
- the *weight* is $w = a_1 + a_2 + a_3$;

- For our diagrams we have $a_r \geq 1$ and the sums converge;
- $C_{a_1,a_2,a_3}(\tau)$ is a modular function under $SL(2,\mathbb{Z})$.

Laplacian on moduli space

- What is the structure of the space of Kronecker-Eisenstein series $C_{a,b,c}(\tau)$?
- Tools : The Laplacian $\Delta = 4\tau_2^2 \partial_\tau \partial_{\bar\tau}$ acts algebraically on the space of $C_{a,b,c}$.

$$\begin{aligned}\Delta C_{a,b,c} &= abC_{a+1,b-1,c} + \frac{1}{2}abC_{a+1,b+1,c-2} - 2abC_{a+1,b,c-1} \\ &\quad + \frac{1}{2}a(a-1)C_{a,b,c} + \text{ 5 permutations of } (a,b,c)\end{aligned}$$

– Δ preserves the "weight" $w = a+b+c$;
– proven by differentiating term by term and using algebraic rearrangements;

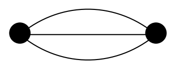
- One of the subscript indices on the right side may equal 0 or -1,

$$\begin{aligned}C_{a,b,0} &= E_a E_b - E_{a+b} && a+b \geq 3 \\ C_{a,b,-1} &= E_{a-1}E_b + E_a E_{b-1} && a,b \geq 2\end{aligned}$$

– all logarithmic divergences of the form E_1 cancel out.

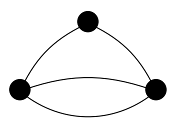
Examples at low weight w

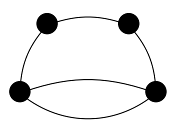
- We find *inhomogeneous Laplace-eigenvalue equations,*

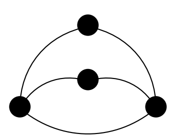
$w = 3$ $\quad C_{1,1,1} =$  $\quad \Delta C_{1,1,1} = 6E_3$

  - Use $\Delta E_3 = 6E_3$ to get $\Delta(C_{1,1,1} - E_3) = 0$;
  - constant determined from asymptotics $C_{1,1,1} = E_3 + \zeta(3)$

    (obtained earlier by Zagier using direct calculation of sums)

$w = 4$ $\quad C_{2,1,1} =$  $\quad (\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2$

$w = 5$ $\quad C_{3,1,1} =$  $\quad (\Delta - 6)C_{3,1,1} = 3C_{2,2,1} + 16E_5 - 4E_2E_3$

$w = 5$ $\quad C_{2,2,1} =$  $\quad \Delta C_{2,2,1} = 8E_5$

  - Note eigenvalues of the form $s(s-1)$ for $s = 1, 2, 3$;

Structure Theorem for $C_{a,b,c}$ modular functions

- $C_{a,b,c}(\tau)$ are linear combinations of modular functions $\mathfrak{C}_{w;s;\mathfrak{p}}(\tau)$ which satisfy

$$\big(\Delta - s(s-1)\big)\mathfrak{C}_{w;s;\mathfrak{p}} = \mathfrak{F}_{w;s;\mathfrak{p}}\big(E_{s'}, \zeta(s'')\big)$$

  - an inhomogeneous eigenvalue equation of weight $w = a+b+c$;
  - $\mathfrak{F}$ is a polynomial of degree 2 in $E_{s'}$ with $2 \leq s' \leq w$;
  - depends on $\zeta(s'')$ for s'' an odd integer $3 \leq s'' \leq w$;

$$s = w - 2\mathfrak{m} \qquad \mathfrak{m} = 1, \cdots, \left[\frac{w-1}{2}\right] \qquad \mathfrak{p} = 0, \cdots, \left[\frac{s-1}{3}\right]$$

- Examples at low weight

$w=3$	$s=1$	$0^{(1)}$
$w=4$	$s=2$	$2^{(1)}$
$w=5$	$s=1,3$	$0^{(1)} \oplus 6^{(1)}$
$w=6$	$s=2,4$	$2^{(1)} \oplus 12^{(2)}$
$w=7$	$s=1,3,5$	$0^{(1)} \oplus 6^{(1)} \oplus 20^{(2)}$
$w=8$	$s=2,4,6$	$2^{(1)} \oplus 12^{(2)} \oplus 30^{(2)}$

The generating function

- There is a natural generating function,

$$\mathcal{W}(t_1, t_2, t_2|\tau) = \sum_{a,b,c=1}^{\infty} t_1^{a-1} t_2^{b-1} t_3^{c-1} C_{a,b,c}(\tau)$$

Summing gives the sunset diagram for three scalars with masses $M_r^2 = -t_r \tau_2$,

$$\mathcal{W}(t_1, t_2, t_2|\tau) = \sum_{(m_r,n_r)\neq(0,0)} \delta_{m,0}\, \delta_{n,0} \prod_{r=1}^{3} \left(\frac{\tau_2}{\pi |m_r \tau + n_r|^2 - t_r \tau_2} \right)$$

- The algebraic representation of the Laplacian induces a differential action on $\mathcal{W}$,

$$\Delta \mathcal{W} - \mathfrak{L}^2 \mathcal{W} = \mathfrak{R}$$

$$\begin{aligned}
\mathfrak{D} &= t_1 \partial_1 + t_2 \partial_2 + t_3 \partial_3 \\
\mathfrak{L}^2 &= \mathfrak{D}^2 + \mathfrak{D} + (t_1^2 + t_2^2 + t_3^2 - 2t_1 t_2 - 2t_2 t_3 - 2t_3 t_1)(\partial_1 \partial_2 + \partial_2 \partial_3 + \partial_3 \partial_1) \\
\mathfrak{R} &= \text{quadratic polynomial in the Eisenstein series } E_s
\end{aligned}$$

Proof via generating function

- Permutation symmetry in (a,b,c) induces permutation symmetry in (t_1,t_2,t_3).
  - $\mathfrak{S}_3$ adapted coordinates,

$$\begin{aligned} u &= t_1+t_2+t_3 & \varepsilon &= e^{2\pi i/3} \\ v/\sqrt{2} &= t_1+\varepsilon t_2+\varepsilon^2 t_3 & (t_1,t_3,t_2)(u,v,\bar v) &= (u,\bar v,v) \\ \bar v/\sqrt{2} &= t_1+\varepsilon^2 t_2+\varepsilon t_3 & (t_2,t_3,t_1)(u,v,\bar v) &= (u,\varepsilon^2 v,\varepsilon \bar v) \end{aligned}$$

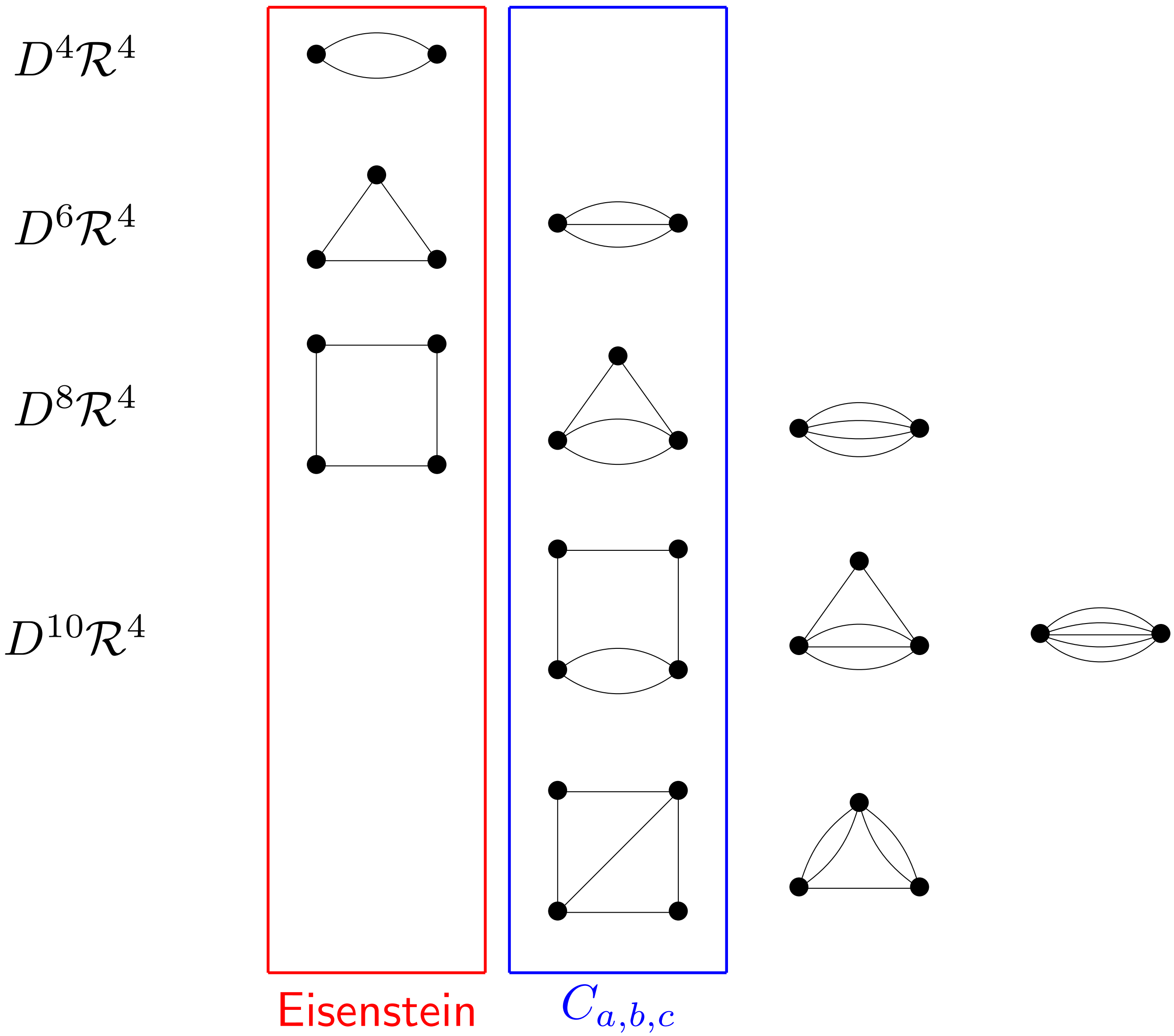
  - $\mathfrak{L}^2=\mathfrak{L}_0^2-\mathfrak{L}_1^2-\mathfrak{L}_2^2$ Casimir of $SO(1,2)$ generated by $\mathfrak{L}_0,\mathfrak{L}_1,\mathfrak{L}_2$;
  - Simultaneously diagonalize the $\mathfrak{S}_3$-invariant operators $\mathfrak{D}$, $\mathfrak{L}_0^2$, and $\mathfrak{L}^2$

$$\begin{aligned} \mathfrak{D}\mathcal{W}_{w;s;\mathfrak{p}} &= w\mathcal{W}_{w;s;\mathfrak{p}} & \mathfrak{D} &= t_1\partial_1+t_2\partial_2+t_3\partial_3 \\ \mathfrak{L}^2\mathcal{W}_{w;s;\mathfrak{p}} &= s(s-1)\mathcal{W}_{w;s;\mathfrak{p}} & \mathfrak{L}^2 &= -(u^2-2v\bar v)(\partial_u^2-2\partial_v\partial_{\bar v}) \\ \mathfrak{L}_0^2\mathcal{W}_{w;s;\mathfrak{p}} &= -9\mathfrak{p}^2\mathcal{W}_{w;s;\mathfrak{p}} & \mathfrak{L}_0 &= iv\partial_v-i\bar v\partial_{\bar v} \end{aligned}$$

  - $\mathfrak{S}_3$-invariance of eigenfunctions requires $\mathfrak{p}$ to be integer;
  - which explains multiplicities $[(s-1)/3]$.

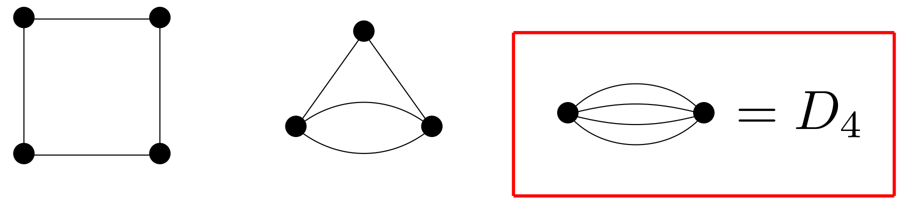
$\Longrightarrow$ constructive proof of Structure Theorem.

Recall ....


$D^4\mathcal{R}^4$
$D^6\mathcal{R}^4$
$D^8\mathcal{R}^4$
$D^{10}\mathcal{R}^4$
Eisenstein
$C_{a,b,c}$

Conjectured relation for modular functions in $D^8\mathcal{R}^4$

- $D^8\mathcal{R}^4$ requires



  - The modular function D_4 is ***not*** of the form $C_{a,b,c}$
  - no useful algebraic representation of the Laplacian is available (yet ?)

- Tools: take an educated guess + check asymptotic behavior near cusp.
  - Relations for $C_{a,b,c}$ involved linear combinations for given weight;
  - Consider combinations of D_4, $C_{2,1,1}$, E_4, and E_2^2

$$(\Delta - 2)\big(D_4 + \alpha C_{2,1,1} + \beta E_2^2 + \gamma E_4\big)$$

- Inspection of asymptotics near the cusp $\tau_2 \to \infty$, leads us to conjecture,

$$D_4 = 24 C_{2,1,1} + 3E_2^2 - 18E_4$$

  - as an *exact relation* between modular functions and Feynman diagrams

- Additional support from direct numerical evaluation of the multiple sums.

Structure of the asymptotics near the cusp

- The expansion near the cusp $\tau_2 \to \infty$ takes the following form,

$$D_4(\tau) = \sum_{k,\bar k=0}^{\infty} \mathcal{D}_4^{(k,\bar k)}(\pi\tau_2)\, q^k \bar q^{\bar k} \qquad\qquad q = e^{2\pi i \tau}$$

- We have checked the following asymptotics (similar asymptotics for $C_{2,1,1}$, E_4, E_2^2)

$$\begin{aligned}
\mathcal{D}_4^{(0,0)}(y) &= \frac{y^4}{945} + \frac{2\zeta(3)y}{3} + \frac{10\zeta(5)}{y} - \frac{3\zeta(3)^2}{y^2} + \frac{9\zeta(7)}{4y^3} \\
\mathcal{D}_4^{(0,1)}(y) &= \frac{4y^2}{15} + \frac{2y}{3} + 2 + \frac{4}{y} + \frac{12\zeta(3)}{y} - \frac{6\zeta(3)}{y^2} + \frac{9}{2y^2} + \frac{9}{4y^3} \\
\mathcal{D}_4^{(1,0)}(y) &= \mathcal{D}_4^{(0,1)}(y)
\end{aligned}$$

How could the conjecture fail ?

- Consider the difference $F = D_4 - 24C_{2,1,1} - 3E_2^2 + 18E_4$
  - the conjecture states $F = 0$

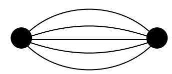
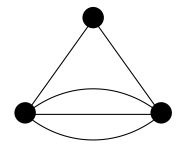
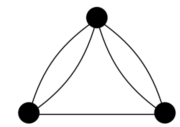
- If the conjecture were to fail, then $F \neq 0$ and its properties are,
  - modular function under $SL(2, \mathbb{Z})$;
  - its pure power part in the expansion near the cusp vanishes;
    - $\Longrightarrow F$ is a *cusp form*
  - Vanishing of leading exponential restricts it further.

- Cusp forms are rather *rare objects*.

- For example, Maass forms are cusp forms that satisfy $\Delta f_s = s(s-1) f_s$;
  - require $s = \frac{1}{2} + i\sigma$ with $\sigma > 13.8$ [communication from Stephen Miller]

- If conjecture fails, then we have an interesting new construction of cusp forms.

Conjectured relation for modular functions in $D^{10}\mathcal{R}^4$

$D^{10}\mathcal{R}^4$ requires $D_5 =$  $D_{3,1,1} =$  $D_{2,2,1} =$ 

in addition to E_5, $C_{3,1,1}$, $E_2 E_3$, and $E_2 C_{1,1,1}$ functions of weight 5.

- Educated guesses and inspection of asymptotics near the cusp lead us to conjecture,

$$
\begin{aligned}
D_5 &= 60 C_{3,1,1} + 10 E_2 C_{1,1,1} - 48 E_5 + 16\zeta(5) \\
40 D_{3,1,1} &= 300 C_{3,1,1} + 120 E_2 E_3 - 276 E_5 + 7\zeta(5) \\
10 D_{2,2,1} &= 20 C_{3,1,1} - 4 E_5 + 3\zeta(5)
\end{aligned}
$$

- We expect this pattern will continue for higher $D^{2w}\mathcal{R}^4$ interactions with $w > 5$.

Genus-one coefficients of $D^8\mathcal{R}^4$ and $D^{10}\mathcal{R}^4$

- Integration over moduli space $\mathcal{M}_1$ produces behavior that is non-analytic in s, t, u;
  - branch cuts due to loops with massless strings for $s, t, u, \ll 1$;
  - non-analytic part may be isolated systematically,
    [ED, Phong 1993; Green, Russo, Vanhove 2008]
  - Analytic part is unique only after non-analytic part has been specified.

- Partition fundamental domain $\mathcal{M}_1$ at fixed large $L \gg 1$
  - $\tau_2 > L$ gives non-analytic contributions in s, t, u;
  - $\tau_2 < L$ gives analytic contributions in s, t, u;

- For compactifications on $\mathbb{T}^d$, for example,

$$\mathcal{E}_{D^8\mathcal{R}^4}(T_d, L) = \frac{1}{2}\int_{\mathcal{M}_1(\tau_2<L)} d\mu_1 \big(\Delta C_{2,1,1} - 5E_4 + E_2^2\big)\, \Gamma_d(T_d|\tau)$$

  - non-analytic parts cancel when comparing different moduli T_d and T'_d;
  - "Differences" produce well-defined and unique analytic parts as $L \to \infty$.

Summary and outlook

- Low energy expansion of string theory has revealed a rich structure of
  - non-holomorphic Kronecker-Eisenstein series on genus-one Riemann surfaces;
  - Zhang-Kawazumi modular invariant on genus-two Riemann surfaces;
  - differential and algebraic interrelations;
  - concrete analytic evaluation of local effective interactions beyond BPS.

- Extensions at genus-one
  - Understand general interrelations of Kronecker-Eisenstein series beyond $C_{a,b,c}$;
  - Identify structure of the ring of all such non-holomorphic modular forms.

- Extensions at genus-two
  - Lifts to toroidal compactifications [Pioline 2015]
  - Differential relations obeyed by higher order generalizations of Zhang-Kawazumi invariants [ED, Green, Vanhove] ... in progress ...