On the generating functions of some sequences of integers defined by the evolution of cellular automaton Rule 150

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Abstract

This is the second of a set of three papers about integer sequences associated with each member of a class of cellular automata, CAs, defined in the first paper. For the case of CA(150), not treated there, this paper provides the extended treatment it calls for. A general treatment of the generating functions, GFs, of its integer sequences is provided, and special attention is given to the special cases of the sequences of numbers $N(t)$, the number of ones in the t-th line of its evolution from an initial state of single seed, and $B(t)$, which is the decimal integer equal to the number of which the t-th line is the binary representation. Proof of Wolfram's formula giving $N(t)$ as a certain product of Jacobsthal numbers, determined by the binary structure of t, is reviewed, because the unified approach followed for such sequences makes clear that an essentially identical factorisation structure holds for the $B(t)$ also. But the factors involved in giving a formula for the $B(t)$ are more complicated than the simple Jacobsthal numbers. In particular the factor for any string of ones bounded by zeros depends, not only on the number of ones in the string, but also on the position of any such string in the binary representation of t. Full detail is given, for the generalisation to the $B(t)$ of Wolfram's formula for the $N(t)$. This constitutes the major achievement of this paper.

1 Introduction

This paper is the second of a series of three papers which study various integer sequences defined by a set Σ of one dimensional cellular automata, CAs, [1, 2], and provides a unified treatment of their generating functions, GFs. The first paper [6] of the series defines the set of CAs Σ, and provides explicit formula for their general GF $G(t, x)$ – see Definition 1, and, for two important special cases, for all CAs $\in \Sigma$, except for one, CA(150), governed by rule number 150 [1]. This has been deferred from [6] to be the sole subject of the present paper, because it needs a much more extensive treatment than does any of the others. It is also the member of Σ for which the most striking new results are found.

For background information on CAs in general and on CA(150) in particular, see the seminal paper [1], the book NKS [2], and the Wolfram Web Resource, Math World [3]. The very useful index to NKS lists more than twenty locations where matters regarding CA(150) are discussed. See [4] for information about generating functions.

The evolution of CA(150) from an initial state of single seed [1] is displayed in Fig. 1. If the lines of the display are indexed by 'times' $t = 0, 1, \dots$, then the display may be used to define various integer sequences. A general treatment of the GFs of such sequences is given here. Special attention is focussed on the two special cases

(i) $\{\mathcal{N} = N(t), \quad t = 0, 1, \dots\}$, where $N(t)$, is the weight of the t-th line, which is just the number of ones it contains, and

(ii) $\{\mathcal{B} = B(t), \quad t = 0, 1, \dots\}$, where $B(t)$ is the decimal integer of which the t-th line is the binary representation.

The values of the first sixteen elements of $\mathcal{N}$ and $\mathcal{B}$ are attached to the display in Fig. 1. Both sequences are well-known and widely studied. The sequence $\mathcal{N}$ features in that invaluable source

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of results, relationships, and insights concerning integer sequences, the On-line Encyclopedia of Integer Sequences [5] as A001045, while the sequence $\mathcal{B}$ is A038184 there. In both cases their relevance to discussion of CA(150) is noted in [5].

Usually displays for Rule 150 are presented in centre-justified format. But here Fig. 1, and the text of all but Section 5, treat a left-justified display. This is convenient because of the wish to deal with GFs, as defined in Definition 1 below, which contain only non-negative powers of x

Let $r = 0, 1, 2, \ldots$, and $s = 0, 1, 2, \ldots$, label the rows and columns of the display shown in Fig. 1, and of extended versions of it.

1	0	1	1
1 1 1	1	3	7
1 . 1 . 1	2	3	21
1 1 . 1 . 1 1	3	5	107
1 . . . 1 . . . 1	4	3	273
1 1 1 . 1 1 1 . 1 1 1	5	9	1911
1 . 1 . . . 1 . . . 1 . 1	6	5	5189
1 1 . 1 1 . 1 1 1 . 1 1 . 1 1	7	11	28123
1 . . . . 1 1	8	3	65793
1 1 1 . . . . . 1 1 1 . . . . . 1 1 1	9	9	460551
1 . 1 . 1 . . . 1 . 1 . 1 . . . 1 . 1 . 1	10	9	1381653
1 1 . 1 . 1 1 . 1 1 . 1 . 1 1 . 1 1 . 1 . 1 1	11	15	7039851
1 . . . 1 . . . . 1 . . . . 1 . . . 1	12	5	17829905
1 1 1 . 1 1 1 . . . . . 1 1 1 . . . . . 1 1 1 . 1 1 1	13	15	124809335
1 . 1 . . . 1 . 1 . . . 1 . 1 . 1 . . . 1 . 1 . . . 1 . 1	14	11	340873541
1 1 . 1 1 . 1 1 . 1 1 . 1 1 . 1 . 1 1 . 1 1 . 1 1 . 1 1 . 1 1	15	21	1840690907

Fig. 1: Display of the evolution of Rule 150.

The entries in the last three columns give t, and the values of $N(t)$, and $B(t)$ values for the t-th line of the display

Definition 1. The generating function of the display shown in Fig. 1 is given by

$$G(t, x) = \sum_{r=0}^{\infty} g_r(x) t^r \quad \text{where} \quad g_r(x) = \sum_{s=0}^{\infty} g_{rs} x^s, \tag{1}$$

and where $g_{rs} \in \{0, 1\}$ is the entry at the intersection of row r and column s of the display.

Definition 2 . The generating function of the sequence $\mathcal{N}$ of integers $N(r)$ is given by

$$G(t, 1) = \sum_{r=0}^{\infty} N(r) t^r \quad \text{where} \quad N(r) = \sum_{s=0}^{\infty} g_{rs}. \tag{2}$$

Definition 3 .The generating function of the sequence of $\mathcal{B}$ of integers $B(r)$ is given by

$$G(t, 2) = \sum_{r=0}^{\infty} B(r) t^r \quad \text{where} \quad B(t) = \sum_{s=0}^{\infty} g_{ts} 2^s. \tag{3}$$

1.1 The general approach and achievements of the paper

In Section 2, inspection of Fig. 1, and more extensive versions of it, enables their self-similarity structure to be formualed by the statement (6) in 3×3 matrix form, in which matrices of size 2^{ω}, $\omega = \omega(k)$, $k = 0, 1, \ldots$, are mapped into matrices of size $2^{2\omega}$. This is then translated in Section 3 into the associated mapping (12) of contributions to $G(t, x)$ corresponding to the matrices just alluded to. By an appropriate, but less than fully intuitive, change of variables, the latter map is replaced by an equivalent 2×2 form – see (17) and (18). The more important of the two types of variables thereby reached, λ_{ω}, corresponds exactly to matrices of the type the $2^{\omega} \times 2^{2\omega}$, the first of which, of size 2×4 coincides with the first 2 lines of Fig. 1. The second one, of size 4×8, coincides with the first four lines of Fig. 1, and so on. It is immediate to pass to a second order difference equation satisfied by the λ variables, which provides a method of sequentially calculating the first $2^{\omega(k)}$ terms of $G(t, x)$ for $k = 2, 3, \ldots$ using the $k = 0, 1$ cases as initial conditions. This secures control, in principle of the self-similarity properties of Fig. 1, for general x, and for $x = 1$ and $x = 2$ in particular. This is done without use at any point of modulo two arithmetic.

1.2 The cases of $x = 1$ and $x = 2$

Section 5 is devoted to the case $x = 1$. Whereas in all but Section 5 use of modulo two arithmetic is avoided, perfectly valid use of it may be made in discussion of self-similarity when $x = 1$. Also there is no need to use a display in left-justified format, and so discussion may be refered to the usual centre-justified display. Here the famous Wolfram formula [1] takes centre stage. It gives a formula quoted in Section 5 for $N(t)$ in terms of Jacobsthal numbers, also defined there. A derivation based on the block-matrix formalism of [7] and [8] is given in [9], leading to a formula (34) for $G(t, 1)$. This is reviewed in Section 5.1, and somewhat extended here, because the Wolfram formula and this method of proving it, generalise to the case of $x = 2$ and the $B(t)$. In Section 5.2, the description of the self-similarity structure of Fig. 1 found in [10], is reformulated and extended to include, a general statement of it, the corresponding second order difference equation, a formula (53) for $G(t, 1)$, and a proof within this approach of Wolfram's formula. There is a certain parallelism in structure but not in detail between the final results in Section 5.1 and 5.2. The results of Section 5.2 also follow by setting $x = 1$ in the general formalism of Section 4

In Section 6, the results of Sections 2 – 4 are specialised to the case of $x = 2$. Versions for $x = 2$ of key equations – (17) to (21) of Section 4 – are displayed. This leads to a formula for $G(t, 2)$. But, whereas the analogous formula for $G(t, 1)$ involves two constant matrices, one of the two matrices in the present case is not constant. This causes significant technical complications, although the factorisation properties of the $N(t)$ epxressed by Wolfram's formula, do survive in the $B(t)$ context, as is shown in Section 6.1. To make progress towards a formula for $B(t)$ valid for all t, it must be shown what are the values of the factors involved. There is one factor for each separate string of ones in the binary representation $t = t_B$ of t. But, in contrast to the case of $N(t)$, such factors depend not only on the length of the string, but now, for $B(t)$, on its position in t_B. Expressions for these factors are obtained in Section 6.2, allowing $B(t)$ to be evaluated for any t. The analogue so obtained for $B(t)$, of Wolfram's formula for $N(t)$, is one of the main results of this paper.

2 The self-similarity structure

To elucidate the self-similarity structure of the display of Fig. 1 for Rule 150, begin with some definitions:

$$a_1 = 1, \quad b_1 = 0, \quad c_1 = 1. \tag{4}$$

$$a_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5}$$

$$
\begin{aligned}
a_4 &= \begin{pmatrix} a_2 & b_2 \\ a_2 & c_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \\
b_4 &= \begin{pmatrix} 0 & 0 \\ c_2 & b_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \\
c_4 &= \begin{pmatrix} a_2 & b_2 \\ b_2 & a_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}.
\end{aligned}
\tag{6}
$$

In (4–6), the subscripts $1, 2, 4$ are used for matrices with

$$\omega(k) = 2^k, \quad k = 0, 1, \ldots, \tag{7}$$

rows and columns. Thus the definitions (5) and (6) conform to the pattern

$$a_{2\omega} = \begin{pmatrix} a & b \\ a & c \end{pmatrix}, \quad b_{2\omega} = \begin{pmatrix} 0 & 0 \\ c & b \end{pmatrix}, \quad c_{2\omega} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \tag{8}$$

where $\omega = \omega(k), \quad k = 0, 1, \ldots,$ and the subscript $\omega(k)$ should be applied to all the unlabeled matrix element entries here.

Fig. 2 was originally drawn up in relation to the 2×2 block structure of Fig. 1, but it can be seen to agree exactly with the structure of Fig. 1 at the 4×4 and 8×8 level, and beyond. Thus no subscripts are shown in Fig. 2

	t
a b	0
a c c b	1
a b a b a b	2
a c b a b a c b	3
a b 0 0 a b 0 0 a b	4
a c c b a c c b a c c b	5
a b a b 0 0 a b 0 0 a b a b	6
a c b a c b a c c b a c b a c b	7

Fig. 2: Block structure of display for Rule 150

Eq. (8) is therefore a full expression of the self-similarity structure of the display of Fig. 1 for cellular automaton Rule 150.

3 Use of the self-similarity result (8)

Turn now to the implications for the generating function (1) of the algebraic statement (8) of the self-similarity of the display for Rule 150, writing

$$\alpha, \beta, \gamma \quad \text{in place of} \quad a, b, c, \tag{9}$$

for the expressions in generating functions that correspond to the matrices of Section 2.

The matrices of (5) carry over into generating function work the expressions

$$\begin{aligned}
\alpha_2(t,x) &= \left\{ \begin{array}{rcr} 1 & + & x.0 \\ +t.1 & + & xt.1 \end{array} \right\} = 1+t+xt \\
\beta_2(t,x) &= \left\{ \begin{array}{rcr} 0 & + & x.0 \\ +t.1 & + & xt.0 \end{array} \right\} = t \\
\gamma_2(t,x) &= \left\{ \begin{array}{rcr} 1 & + & x.0 \\ +t.0 & + & xt.1 \end{array} \right\} = 1+xt,
\end{aligned} \tag{10}$$

whereas those of (6) carry the expressions

$$\begin{aligned}
\alpha_4(t,x) &= \left\{ \begin{array}{rcr} \alpha_2(t,x) & + & x^2\beta_2(t,x) \\ +t^2\alpha_2(t,x) & + & x^2t^2\gamma_2(t,x) \end{array} \right\} \\
&= 1+t+t^2+t^3+x(t+t^3)+x^2(t+t^2)+x^3t^3 \\
\beta_4(t,x) &= \left\{ \begin{array}{rcr} 0 & + & x^2.0 \\ +t^2\gamma_2(t,x) & + & x^2t^2\beta_2(t,x) \end{array} \right\} \\
&= t^2(1+xt+x^2t) \\
\gamma_4(t,x) &= \left\{ \begin{array}{rcr} \alpha_2(t,x) & + & x^2\beta_2(t,x) \\ +t^2\beta_2(t,x) & + & x^2t^2\alpha_2(t,x) \end{array} \right\} \\
&= 1+t+t^3+xt+x^2(t+t^2+t^3)+x^3t^3,
\end{aligned} \tag{11}$$

The entries within braces in (10) and (11) serve to highlight the role of (8) in the sequential construction of the α, β, γ variables.

The actual contribution from such expressions to the display to $G(t,x)$ depends further on the particular place the corresponding matrix occupies in the display. This requires that it is multiplied by the product of appropriate powers of t and of x. If the top left-hand element of a display block resides at position (r,s), then this power is $t^r x^s$. Use of this *within* the 4×4 blocks corresponding to the expressions in (11) can be seen in the explicit powers of x and t in its second, fourth and sixth lines.

The translation of the self-similarity statement of (8) for display matrix blocks into the generating function context can therefore be epxressed as

$$\begin{pmatrix} \alpha_{2\omega} \\ \beta_{2\omega} \\ \gamma_{2\omega} \end{pmatrix} = \begin{pmatrix} 1+t^\omega & x^\omega & (xt)^\omega \\ 0 & (xt)^\omega & t^\omega \\ 1+(xt)^\omega & x^\omega + t^\omega & 0 \end{pmatrix} \begin{pmatrix} \alpha_\omega \\ \beta_\omega \\ \gamma_\omega, \end{pmatrix}, \tag{12}$$

where $\omega = \omega(k) = 2^k$, for $k = 0,1,2,\ldots$.

This can also be written in the form

$$\begin{pmatrix} \alpha_{2\omega} \\ \beta_{2\omega} \\ \gamma_{2\omega} \end{pmatrix} = [\mathrm{A}_0(x^\omega) + t^\omega \mathrm{A}_1(x^\omega)] \begin{pmatrix} \alpha_\omega \\ \beta_\omega \\ \gamma_\omega, \end{pmatrix}, \tag{13}$$

where the matrices $\mathrm{A}_0(y)$ and $\mathrm{A}_1(y)$ are defined by

$$\mathrm{A}_0(y) = \begin{pmatrix} 1 & y & 0 \\ 0 & 0 & 0 \\ 1 & y & 0 \end{pmatrix}, \quad \mathrm{A}_1(y) \begin{pmatrix} 1 & 0 & y \\ 0 & y & 1 \\ y & 1 & 0 \end{pmatrix}. \tag{14}$$

4 Towards the generating function $G_{150}(t,x)$

Eq. (13) does not provide the final form of the self-similarity structure of the display of Fig. 1. A natural variable for use in place of α_ω is

$$\lambda_\omega = \alpha_\omega + x^\omega \beta_\omega. \tag{15}$$

This expressions yields the contribution of the first 2^ω of Fig. 1 to $G_{150}(t,x)$, when used with its top-left element at the site with $r=s=0$. Checking this for $k=2$, it is seen that λ_4 corresponds to the 4×8 matrix of the first four lines of Fig.1. Replacing α_ω by λ_ω throughout (13), motivates the introduction, in place of the variable γ_ω, of the variable

$$\mu_\omega = \gamma_\omega + (x^\omega - 1)\beta_\omega. \tag{16}$$

This step emerges directly from the algebra, but is not an intuitively obvious one. While the quantity $\gamma_\omega + x^\omega\beta_\omega$, for example in the case $k=2$, does correspond to a 4×8 matrix of frequent occurrence in Fig. 1, the need for the correction term in (16) is not a priori clear. But it is *unambiguously* called for in order to realise the following important algebraic statement of the self-similarity structure of Fig. 1

$$\begin{aligned} \lambda_{2\omega} &= (1+t^\omega)\lambda_\omega + x^\omega t^\omega (1+x^\omega)\mu_\omega \\ \mu_{2\omega} &= (1+x^\omega t^\omega)\lambda_\omega + (x^{2\omega}-1)t^\omega\mu_\omega,, \end{aligned} \tag{17}$$

or

$$\begin{pmatrix} \lambda_{2\omega} \\ \mu_{2\omega} \end{pmatrix} = [\mathrm{C}_0 + t^\omega \mathrm{C}_1(x^\omega)] \begin{pmatrix} \lambda_\omega \\ \mu_\omega \end{pmatrix}, \tag{18}$$

where

$$\mathrm{C}_0 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{C}_1(y) = \begin{pmatrix} 1 & y(y+1) \\ y & y^2-1 \end{pmatrix}. \tag{19}$$

While the variable β_ω has been marginalised in the discussion, it may be noted that

$$\beta_{2\omega} = t^\omega(\mu_\omega + \beta_\omega). \tag{20}$$

Eq. (17) is clearly important. It shows that the λ and μ variables describe the self-similarity properties of Fig. 1 and $G_{150}(t,x)$ in 2×2 matrix form. Further it is easy to deduce the second order difference equations satisfied by the λ variables and by the μ variables. The latter leads directly, via (20), to the third order difference equation satisfied by the β variables. It seems reasonable to expect that the α variables also satisfy a third order difference equation, but this has not been confirmed. Nor has the analogous opinion for the γ variables been examined.

A very useful second order difference equation satisfied by the λ variables follows from (17)

$$\begin{aligned} \lambda_{4\omega} &= \left[1 + t^{2\omega}(1 - x^\omega + x^{2\omega} - x^{3\omega} + x^{4\omega})\right]\lambda_{2\omega} \\ &\quad + x^\omega t^{2\omega}(1+x^{2\omega})\left[1 + t^\omega(1 - x^\omega + x^{2\omega})\right]\lambda_\omega, \end{aligned} \tag{21}$$

where $\omega = \omega(k), \quad k = 0,1,2,\ldots$. Its initial conditions are

$$\lambda_1 = \alpha_1 = 1, \tag{22}$$

$$\lambda_2 = \alpha_1 + x^2\beta_1 = 1 + t(1+x+x^2). \tag{23}$$

It can be easily checked that the case

$$\lambda_4 = \left[1 + t^2(1 - x + x^2 - x^3 + x^4\right]\lambda_2 + xt^2(1+x^2)\left[1 + t(1 - x + x^2)\right]\lambda_1, \tag{24}$$

does indeed yield the first four terms of $G_{150}(t,x)$.

In fact an exact formula for $G_{150}(t,x)$ follows from (18). To realise it, write

$$\Gamma_{\omega(k)} = \mathrm{C}_0 + t^{\omega(k)}\mathrm{C}_1(x^{\omega(k)}), \tag{25}$$

and

$$\lambda_{\omega(r)} = \begin{pmatrix} 1 & 0 \end{pmatrix} \prod_{k=0}^{r-1} \Gamma_{\omega(k)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{26}$$

in which the ordering, $k = 0,1,2,\ldots,r-1$, from right to left must be retained. This gives $G_{150}(t,x)$ as far as its term in t^{r-1}. Formally considering the limit $\lim_{r\to\infty}$, it is seen that $G_{150}(t,x)$ is given by a result with an infinite continued product structure.

5 The special case of $G_{150}(t,1)$

When $x = 1$, there is no reason for preferring the left-justified display of Fig. 1 over the more familiar centre-justified one, so in this section, the discussion may be referred to the latter.

The generating function of interest here is $G_{150}(t,1)$, which involves the weights $N(t)$ of the lines of Fig. 1, whether left or centre-justified. These are governed by Wolfram's formula [1], given in terms of the Jacobsthal numbers $\{\chi_n, \quad n = 0, 1, \dots\}$, *i.e.* sequence A001045 of [5], defined by

$$\chi_{n+2} = \chi_{n+1} + 2\chi_n, \quad \chi_0 = 1, \quad \chi_1 = 3. \tag{27}$$

It follows from (27) that

$$\chi_n = \frac{1}{3}(2^{n+2} + (-1)^{n+1}). \tag{28}$$

The formula in question depends on the binary representation t_B of t. Each string 1^r of ones bounded by zeros which occurs in t_B contributes a factor χ_r to $B(t)$. Thus, if

$$t_B = 1^{r_p}0^{a_p} \dots 1^{r_2}0^{a_2}1^{r_1}0^{a_1}, \tag{29}$$

with the most significant bit listed first, then Wolfram's formula states that

$$N(t) = \prod_{j=1}^{p} \chi_{r_j}. \tag{30}$$

5.1 The n-block approach, and Wolfram's formula

The powerful n-block method of [7] and [8] is of very general applicability. It can evaluate the weights of the lines of the evolution, not only of Rule 150, but also of Rule 90 and of others. Further it can cope with initial states more complicated than single seed ones. However for the case of interest here, use of 2-blocks suffices.

The 2-block view of the display line at time t regards it as a sequence of overlapping 2-blocks

$$\{0,0\}, \{0,1\}, \{1,0\}, \{1,1\}.$$

The first of these plays no part in counting the number $N(t)$ of ones in the line, and the next two play identical roles, so that the counting vector c is given by $c^T = (1,2)$. The display line at $t = 0$ $\dots 00100 \dots$ involves each of the two 2-blocks $\{0,1\}, \{1,0\}$ once, and the 2-block $\{1,1\}$ not at all, so that the initial vector is $u, \quad u^T = (2,0)$. The 2-block approach uses the definition of Rule 150 with modulo two arithmetic to identify the matrices

$$\mathrm{A}_0 = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{A}_1 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \tag{31}$$

which connect the block content of the state of the (centre-justified) display line for time t to the states of the display lines for times $2t$ and $2t+1$ repectively. Thus, if at time t, the display line involves a_1 blocks of types $\{0,1\}$ or $\{1,0\}$, and a_2 of type $\{1,1\}$, then the corresponding values at times $2t$ and $2t+1$ are given by

$$N(2t) = \frac{1}{2}c^T\mathrm{A}_0 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \text{and} \quad N(2t+1) = \frac{1}{2}c^T\mathrm{A}_1 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{32}$$

The factor $\frac{1}{2}$ is required because of the double counting inherent in the 2-block method. Thus, for example,

$$\frac{1}{2}c^T(\mathrm{A}_0 + t^4\mathrm{A}_1)(\mathrm{A}_0 + t^2\mathrm{A}_1)(\mathrm{A}_0 + t\mathrm{A}_1)u, \tag{33}$$

yields the first eight terms of $G_{150}(t,2)$, and reveals the relationship of the t^k term, $k = 0, 1, \dots, 7$ to the binary representation of k.

Further

$$G_{150}(t,1) = \frac{1}{2}c^T \prod_{r=0}^{\infty}(\mathrm{A}_0 + t^{\omega(r)}\mathrm{A}_1)u, \tag{34}$$

with the understanding that the factors of the product are in order $r = 0, 1, \ldots,$ from right to left.

Expansion of (34) as a power series in t make it clear that the subscripts on the the matrices for line t constitute the binary representation t_B of t with most significant bit in the left-most position. If

$$t = \sum_{\alpha=1}^{p} i_\alpha 2^{\alpha-1}, \quad i_\alpha \in \{0,1\}, \quad \alpha = 1, \ldots, p, \tag{35}$$

then

$$t_B = i_p \ldots i_2 i_1, \tag{36}$$

and

$$N(t) = \frac{1}{2}c^T \mathrm{A}_{i_p} \ldots \mathrm{A}_{i_2}\mathrm{A}_{i_1}\ u. \tag{37}$$

Now various facts may be noted

$$(\mathrm{A}_0)^2 = \mathrm{A}_0, \quad \mathrm{A}_0 u = u, \quad c^T \mathrm{A}_0 = c^T, \tag{38}$$

so that factors A_0 at either end of t_B may be ignored, and interior strings of A_0 matrices may be replaced by a single matrix A_0. Since

$$\mathrm{A}_0 = \frac{1}{2}uc^T, \tag{39}$$

it follows that the result for any $N(t)$ can be written as the product of factors of the type

$$H(r) = \frac{1}{2}c^T (A_1)^r u. \tag{40}$$

The first main ingredient – factorisation – of Wolfram's formula has thus emerged. To realise the second one, note that

$$\frac{1}{2}c^T u = 1, \quad \frac{1}{2}c^T (A_1)u = 3, \tag{41}$$

and also that

$$(\mathrm{A}_1)^2 = \mathrm{A}_1 + 2. \tag{42}$$

The identification

$$H(r) = \chi_r, \quad r = 0, 1, \ldots\ , \tag{43}$$

follows directly from these results, so that the full content of Wolfram's formula for the $N(t)$ is elucidated. One example: for $t = 13$, $t_B = 1101$ and

$$N(13) = \frac{1}{2}c^T \mathrm{A}_1\mathrm{A}_1\mathrm{A}_0\mathrm{A}_1 u = \frac{1}{2}c^T \mathrm{A}_1\mathrm{A}_1 u\ \frac{1}{2}c^T \mathrm{A}_1 u = \chi_2\chi_1 = 5 \times 3 = 15. \tag{44}$$

The content of this section extends the derivation in [9] to obtain $G_{150}(t,1)$ in the from (34). It may be stressed at this point that, in the treatment in Section 6, of $G_{150}(t,2)$, the factorisation of the results for $B(t)$ follows an pattern identical to that seen for the $N(t)$, but leads to a more complex final statement than Wolfram's formula does. The simplicity of the latter depends on the fact that *both* of the matrices $\mathrm{A}_0, \mathrm{A}_1$ of (31) are constant.

5.2 The self-smilarity approach to Wolfram's formula

Another approach to the evaluation of the $N(t)$, as defined as Definition 2, is described in [10]. This is based on a derivation of the self-similarity properties of the display of Fig.1 in centre-justified form valid when modulo two arithmetic is used. What is presented here is a polynomial version of the underlying method, rather than the version given in [10], which involves block matrices of ever increasing size.

The derivation of the self-similarity relation for the context of Fig. 1 in [10] need not be repeated here. It leads naturally to the definition of pairs of variables y_k and z_k, $\quad k = 1, 2, \ldots,$ where the notation here is related to the notation there [10] by

$$y_{k+1} = Y_k, \quad z_{k+1} = Z_k, \quad k = 1, 2, \ldots, \tag{45}$$

and is necessary to allow comparison with the work of Section 2 – 4. The values $y_1 = 1, z_1 = 3$ are initial conditions, and results given in [10] are translated here into the form

$$\begin{pmatrix} y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 3+2t & t \end{pmatrix} \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, \tag{46}$$

$$\begin{pmatrix} y_4 \\ z_4 \end{pmatrix} = \begin{pmatrix} 1 & t^2 \\ 3+2t^2 & t^2 \end{pmatrix} \begin{pmatrix} y_2 \\ z_2 \end{pmatrix}, \tag{47}$$

which can be written in general form

$$\begin{aligned} \begin{pmatrix} y_{2\omega} \\ z_{2\omega} \end{pmatrix} &= \begin{pmatrix} 1 & t^\omega \\ 3+2t^\omega & t^\omega \end{pmatrix} \begin{pmatrix} y_\omega \\ z_\omega \end{pmatrix} && (48) \\ &= (\mathrm{D}_0 + t^\omega \mathrm{D}_1) \begin{pmatrix} y_\omega \\ z_\omega \end{pmatrix} && (49) \\ &= \prod_{k=0}^{r} (\mathrm{D}_0 + t^{\omega(k)} \mathrm{D}_1) \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, && (50) \end{aligned}$$

with $\omega = \omega(k)$, and

$$\mathrm{D}_0 = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}, \quad \mathrm{D}_1 = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}. \tag{51}$$

It is easy to verify that

$$\begin{aligned} y_2 &= 1 + 3t \\ y_4 &= 1 + 3t + 3t^2 + 5t^3 \\ y_8 &= 1 + 3t + 3t^2 + 5t^3 + 3t^4 + 9t^5 + 5t^6 + 11t^7, \end{aligned} \tag{52}$$

so that the first eight terms of $G_{150}(t, 1)$ take very well-known form. Hence writing the initial vector u_1, $\quad {u_1}^T = (1, 3)$, and the 'counting vector' c_1, $\quad {c_1}^T = (1, 0)$, there follows the result

$$G_{150}(t, 1) = {c_1}^T \prod_{k=0}^{\infty} \left[\mathrm{D}_0 + t^{\omega(k)} \mathrm{D}_1 \right] u_1, \tag{53}$$

in which the factors with k-values $0, 1, \ldots,$ occur in that order reading from right to left.

Eqs. (50) and (53) are new here, as also is the following second order difference equation satisfied by the y variables

$$y_{4\omega} = (1 + t^{2\omega} y_{2\omega}) + 2t^{2\omega}(1 + t^\omega) y_\omega, \tag{54}$$

where $\omega = \omega(k)$, $\quad k = 0, 1, \ldots.$ Eq. (54) is a consequence of (48). It is easy to check that, with the initial conditions $y_1 = 1, y_2 = 1 + 3t$, the results for y_4 and y_8 quoted in (52) arise from (54) too.

A proof of Wolfram's formula [1] expressing the $N(t)$ in terms of Jacobsthal numbers can be given in this formalism too. For this purpose, it is necessary only to note that

$$(\mathrm{D}_0)^2 = \mathrm{D}_0, \quad \mathrm{D}_0\, u_1 = u_1, \quad {c_1}^T\, \mathrm{D}_0 = {c_1}^T, \quad \mathrm{D}_0 = u_1 {c_1}^T, \tag{55}$$

so that the derivation goes through in identical fashion to the one given in Section 5.2, differing from it only superficially in detail, but perhaps amusingly so.

The derivation of Wolfram's formula in this section is somewhat easier to complete than that given in Section 5.1, because the self-similarity method of [10] is a simpler, but less powerful and less flexible method than the block-matrix formalism of Willson and Takahashi [7] [8] provides. It is simpler because it directly addresses evolution from the single seed initial state, whereas the block-matrix method is designed to allow treatment of any initial state of CA(150).

In Section 4, the arguments t, x of $\lambda_\omega = \lambda_\omega(t,x)$ were left implicit. However, the difference equation (21) and its initial conditions (22) and (23), when written for $x = 1$, *i.e.* for $\lambda_\omega(t,1)$, coincide with those just obtained for y_ω, so that

$$\lambda_\omega(t,1) = y_\omega. \tag{56}$$

6 Treatment of $G_{150}(t,2)$

6.1 Factorisation of the $B(t)$

This section examines the work of Sections 2 – 4 when $x = 2$, since in this case $G_{150}(t,2)$ generates the numbers $B(t)$ of Definition 3.

When $x = 2$, (15), (16) and (18) read as

$$\lambda_\omega = \alpha_\omega + 2^\omega \beta_\omega, \quad \mu_\omega = \gamma_\omega + (2^\omega - 1)\beta_\omega, \tag{57}$$

$$\begin{pmatrix} \lambda_{2\omega} \\ \mu_{2\omega} \end{pmatrix} = [\mathrm{C}_0 + t^\omega \mathrm{C}_1(2^\omega)] \begin{pmatrix} \lambda_\omega \\ \mu_\omega \end{pmatrix}, \tag{58}$$

where C_0 and C_1 are given by (19) with $y = 2^\omega$, so that

$$\mathrm{C}_1(2^{\omega(0)}) = \begin{pmatrix} 1 & 6 \\ 2 & 3 \end{pmatrix}, \quad \mathrm{C}_1(2^{\omega(1)}) = \begin{pmatrix} 1 & 20 \\ 4 & 15 \end{pmatrix}, \quad \mathrm{C}_1(2^{\omega(2)}) = \begin{pmatrix} 1 & 272 \\ 16 & 255 \end{pmatrix}. \tag{59}$$

With initial vector is u, $\quad u^T = (\lambda_1, \mu_1) = (1,\ 1)$, and counting vector c, $\quad c^T = (1,\ 0)$, it is seen that

$$G_{150}(t,2) = c^T \prod_{k=0}^{\infty} \left[\mathrm{C}_0 + t^{\omega(k)} \mathrm{C}_1(2^{\omega(k)})\right] u. \tag{60}$$

Consider the truncation of (60)

$$\sum_{j=0}^{\omega(r)-1} t^j B(j) = c^T \prod_{k=0}^{r-1} \left[\mathrm{C}_0 + t^{\omega(k)} \mathrm{C}_1(2^{\omega(k)})\right] u. \tag{61}$$

Here the use of c^T yields the value of λ_ω as a polynomial of degree $\omega(r) - 1$ in t, and thereby generates the the first $\omega(r)$ terms of $G_{150}(t,2)$.

Each $B(j)$ defined by (61) contains a product of r matrices with arguments $2^{\omega(k)}$ ordered by $k = 0, 1, \ldots, r-1$, from right to left. Of course the C_0 are constants, independent of any argument; each one merely marks a place in the product at which k changes by one. Further each $B(j)$ is given by the term of the product in which the subscripts on the C-matrices read from left to right give the binary representation j_B of j, as defined by (36).

The results

$$(\mathrm{C}_0)^2 = \mathrm{C}_0, \quad \mathrm{C}_0 u = u, \quad c^T \mathrm{C}_0 = c^T, \quad \mathrm{C}_0 = uc^T, \tag{62}$$

exact analogues of (38) and (39) are true here. It follows from (62) that the factorisation structure of the integers $B(t)$ is exactly the same as of the weights $N(t)$, as described in Wolfram's formula. In fact (62) indicates that the value of any $B(t)$ is determined by the separate strings of ones in the binary representation t_B of t. There is one factor for each such string which depends, here for $x = 2$, not only on its length, but also on its position in t_B. This is due to the fact that the matrix C_1 is not constant, which is in contrast to the situation for the $N(t)$, where the factor χ_n for a string of n ones depends only on n, because the matrix A_1 of (31) is constant.

A simple example offers a good illustration and indicates what more needs to be done to obtain a formula for $B(t)$ valid for all t. Using square brackets to denote the binary representation t_B of t, eq. (61) gives rise easily to

$$
\begin{aligned}
B(1) &= B\,[1] = c^T \mathrm{C}_1(2^{\omega(0)})u && (63)\\
B(4) &= B\,[100] = c^T \mathrm{C}_1(2^{\omega(2)})u && (64)\\
B(5) &= B\,[101] = c^T \mathrm{C}_1(2^{\omega(2)})\mathrm{C}_0\mathrm{C}_1(2^{\omega(0)})u && (65)\\
&= c^T \mathrm{C}_1(2^{\omega(2)})u\; c^T \mathrm{C}_1(2^{\omega(0)})u, && (66)
\end{aligned}
$$

and (66) expresses the factorisation

$$
\begin{aligned}
B(5) &= B(4)B(1) && (67)\\
B\,[101] &= B\,[100]\,B\,[1]\,. && (68)
\end{aligned}
$$

It is evident that this works in general, *e.g.*

$$
B\,[1101001] = B\left[110^5\right] B\left[10^3\right] B\,[1]\,. \qquad (69)
$$

What needs to be done now is the evaluation of $B\left[1^p 0^k\right]$ for $p = 1, 2, \ldots,$ and $k = 0, 1, \ldots.$

6.2 Towards the formula for the integers $B(t)$

It may be helpful to present first a table of the sort of results available on the basis of the work of the last and the present sections. Recall that the most significant bit of t_B is listed first.

$$
\begin{aligned}
B(0) &= 1 \\
B(1) &= B[1] = 7 \\
B(2) &= B[10] = 21 \\
B(3) &= B[11] = 107 \\
B(4) &= B[100] = 273 \\
B(5) &= B[101] = B[100]\,B[1] = B(4)B(1) = 1911 \\
B(6) &= B[110] = 5189 \\
B(7) &= B[111] = 28123 \\
B(8) &= B[1000] = 65793 \\
B(9) &= B[1001] = B[1000]\,B[1] = B(8)B(1) \\
B(10) &= B[1010] = B[1000]\,B[10] = B(8)B(2) \\
B(11) &= B[1011] = B[1000]\,B[11] = B(8)B(3) \\
B(12) &= B[1100] = 17829905 \\
B(13) &= B[1101] = B[1100]\,B[1] = B(12)B(1) \\
B(14) &= B[1110] = 340873541 \\
B(15) &= B[1111] = 1840690907 \\
B(16) &= B[10000] = 4295032833 \\
B(17) &= B[10001] = B[10000]\,B[1] = B(16)B(1) \\
B(18) &= B[10010] = B[10000]\,B[10] = B(16)B(2) \\
B(19) &= B[10011] = B[10000]\,B[11] = B(16)B(3) \\
B(20) &= B[10100] = B[10000]\,B[100] = B(16)B(4) \\
B(21) &= B[10101] = B[10000]\,B[100]\,B[1] = B(16)B(4)B(1) \\
B(22) &= B[10110] = B[10000]\,B[110] = B(16)B(6) \\
B(23) &= B[10111] = B[10000]\,B[111] = B(16)B(7) \\
B(24) &= B[11000] = 282574505115905 \\
B(25) &= B[11001] = B[11000]\,B[1] = B(24)B(1) \\
B(26) &= B[11010] = B[11000]\,B[10] = B(24)B(2) \\
B(27) &= B[11011] = B[11000]\,B[11] = B(24)B(3) \\
B(28) &= B[11100] = 76579889943220241 \\
B(29) &= B[11101] = B[11100]\,B[1] = B(28)B(1) \\
B(30) &= B[11110] = 1464027307232743749 \\
B(31) &= B[11111] = 7905747460774803163 \\
B(32) &= B\left[10^5\right] = 18446744078004518913 \qquad (70)
\end{aligned}
$$

Direct calculations lead to the results

$$
\begin{aligned}
B\left[10^k\right] &= c^T \mathrm{C}_1(2^{\omega(k)})u = 1 + y + y^2 && (71) \\
B\left[110^k\right] &= c^T \mathrm{C}_1(2^{\omega(k+1)})\mathrm{C}_1(2^{\omega(k)})u = 1 + y + y^3 + y^5 + y^6 && (72) \\
B\left[1110^k\right] &= c^T \mathrm{C}_1(2^{\omega(k+2)})\mathrm{C}_1(2^{\omega(k+1)})\mathrm{C}_1(2^{\omega(k)})u \\
&= 1 + y + y^3 + y^4 + y^6 + y^7 + y^8 + y^{10} + y^{11} + y^{13} + y^{14}, && (73)
\end{aligned}
$$

for $y = 2^{\omega(k)}$, $\quad k = 0, 1, \ldots$. These results are the analogues of (40) for $r = 1, 2, 3$, given in general enough form to handle strings of ones of these lengths regardless of the position of their right-most C_1 matrix in in relation to the binary representation of t.

Thus (71) applies to t values $t = 1, 2, 4, \ldots$, giving

$$B(1) = 7, B(2) = 21, B(4) = 273, B(8) = 65793. \tag{74}$$

Similarly (72) applies to t values $3, 6, 12, \ldots$, giving

$$B(3) = 107, B(6) = 5189, B(12) = 17829905 \tag{75}$$

and (73) applies to t values $7, 14, \ldots$, so that

$$B(7) = 28123, B(14) = 340873541. \tag{76}$$

Thus a value can be found for all $B(t)$ for $t < 15$, as well as plenty others.

Treatment of $B(15) = B\left[1^4\right]$, and $B\left[1^p0^k\right]$, $p \geq 4$, requires a general result of which (71 – 73) are the $p = 1, 2$ and 3 examples. To obtain it, note that the sets of coefficients of powers of y in (71 – 73) are given by

$$\begin{array}{c} 1\ 1\ 1 \\ 1\ 1\ 0\ 1\ 0\ 1\ 1 \\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1 \end{array} \quad . \tag{77}$$

These coincide with lines $1, 3, 7$ of the display, and suggest line 15 of the display

$$(1\ 1\ 0)^5\ 1\ (0\ 1\ 1)^5 \tag{78}$$

gives the sequence of coefficients of the required polynomial of degree 30 in $y = 2^\omega$ for $B\left[1^40^k\right]$, a conjecture which works for $B(15) = B\left[1^4\right]$ when $y = 2^{\omega(0)} = 2$, and also for $B(30) = B\left[1^40\right]$ when $y = 2^{\omega(1)} = 4$. The conjecture can of course be proved to be correct in the same way as (71 –73) are proved. These observations give rise to the working hypothesis that the explicit expression sought for $B\left[1^p0^k\right]$ is obtained in the same manner from the line of the display, extended if necessary, for $t = 2^p - 1$. The pattern of the required results can be seen from

$$\begin{array}{ll} p = 1: & (110)^0\ 1^3\ (011)^0 \\ p = 2: & (110)^1\ 1\ (011)^1 \\ p = 3: & (110)^2\ 1^3\ (011)^2 \\ p = 4: & (110)^5\ 1\ (011)^{1+4} \\ p = 5: & (110)^{10}\ 1^3\ (011)^{2+8} \\ p = 6: & (110)^{21}\ 1\ (011)^{5+16} \\ p = 7: & (110)^{42}\ 1^3\ (011)^{10+32}, \end{array} \tag{79}$$

etc., which give the coefficients for the polynomials of degrees 2, 6, 14, 30, 62, 126, 254, etc., in $y = 2^{\omega(k)}$ for the corresponding $B\left[1^p0^k\right]$.

The sequence 1, 5, 21, 85, 341, ..., arising from the even p lines of (79) is governed by the difference equation

$$u_{2r+2} = 4u_{2r} + 1, \quad r = 1, 2, \ldots\ ,$$

with initial condition $u_2 = 1$ and solution given by $3u_{2r} = 2^{2r} - 1$. It is found in [5] as A002450. Also $u_{2r+1} = 2u_{2r}, \quad r = 1, 2, \ldots$, holds for the odd p lines of (79).

Taking (79), which, as far as has been checked, gives rise to correct results, as a given, a formula for $B(t)$ for any t is available, because it is known how to treat explicitly cases such as

$$B\left[1^a0^b1^c0^d\right] = B\left[1^a0^{b+c+d}\right] B\left[1^c0^d\right] \tag{80}$$

$$B\left[1^a0^b1^c0^d1^e0^f\right] = B\left[1^a0^{b+c+d+e+f}\right] B\left[1^c0^{d+e+f}\right] B\left[1^e0^f\right], \tag{81}$$

etc. The obvious ponderous result for arbitrary t_B need not be displayed.

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