3P1b **Quantum Field Theory: Example Sheet 2** Michaelmas 2018

Corrections and suggestions should be emailed to `B.C.Allanach@damtp.cam.ac.uk`*. Starred questions may if you wish be handed in to your supervisor for feedback prior to the class.*

1. Consider a real scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \tag{1}$$

Show that, after normal ordering, the conserved four-momentum $P^\mu = \int d^3x \; T^{0\mu}$ takes the operator form

$$P^\mu = \int \frac{d^3p}{(2\pi)^3} p^\mu a^\dagger_{\vec{p}} a_{\vec{p}} \tag{2}$$

where $p^0 = E_{\vec{p}}$ in this expression. From Eq. (2), verify that if $\phi(x)$ is now in the Heisenberg picture, then

$$[P^\mu, \phi(x)] = -i\partial^\mu\phi(x).$$

2*. Show that in the Heisenberg picture,

$$\dot{\phi}(x) = i[H, \phi(x)] = \pi(x) \quad \text{and} \quad \dot{\pi}(x) = i[H, \pi(x)] = \nabla^2\phi(x) - m^2\phi(x)\,.$$

Hence show that the operator $\phi(x)$ satisfies the Klein-Gordon equation.

3. Let $\phi(x)$ be a real scalar field in the Heisenberg picture. Show that the relativistically normalised states $|p\rangle = \sqrt{2E_{\vec{p}}}a^\dagger_{\vec{p}}|0\rangle$ satisfy

$$\langle 0|\,\phi(x)\,|p\rangle = e^{-ip\cdot x}.$$

4*. In Example Sheet 1, you showed that the classical angular momentum of a field is given by

$$Q_i = \frac{1}{2}\epsilon_{ijk}\int d^3x \;\left(x^j T^{0k} - x^k T^{0j}\right).$$

Write down the explicit form of the angular momentum for a free real scalar field with Lagrangian as in Eq.(1). Show that, after normal ordering, the quantum operator Q_i can be written as

$$Q_i = -\frac{i}{2}\epsilon_{ijk}\int \frac{d^3p}{(2\pi)^3} a^\dagger_{\vec{p}}\left(p^j\frac{\partial}{\partial p_k} - p^k\frac{\partial}{\partial p_j}\right)a_{\vec{p}}.$$

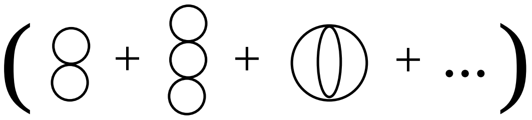
Hence confirm that the quanta of the scalar field have spin zero (i.e. a one-particle state $|\vec{p}\rangle$ has zero angular momentum in its rest frame).

5. Show that the time ordered product $T(\phi(x_1)\phi(x_2))$ and the normal ordered product $:\phi(x_1)\phi(x_2):$ are both symmetric under the interchange of x_1 and x_2. Deduce that the Feynman propagator $\Delta_F(x_1 - x_2)$ has the same symmetry property.

6. Verify Wick's theorem for the case of three scalar fields:

$$T(\phi(x_1)\phi(x_2)\phi(x_3)) \;=\; :\phi(x_1)\phi(x_2)\phi(x_3)): +\phi(x_1)\Delta_F(x_2-x_3) \\ +\phi(x_2)\Delta_F(x_3-x_1)+\phi(x_3)\Delta_F(x_1-x_2).$$

7. Examine $\langle 0|\, S\, |0\rangle$ to order λ^2 in ϕ^4 theory. Identify the different contributions arising from an application of Wick's theorem. Confirm that to order λ^2, the combinatoric factors work out so that the vacuum to vacuum amplitude is given by the exponential of the sum of distinct vacuum bubble types,

$\langle 0|\, S\, |0\rangle = \exp$ 

8. Consider the Lagrangian density for three scalar fields ϕ_i, $i=1,2,3$, given by

$$\mathcal{L} = -\sum_{i=1}^{3}\frac{1}{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i) - \frac{1}{2}m^2(\sum_{i=1}^{3}\phi_i^2) - \frac{\lambda}{8}(\sum_{i=1}^{3}\phi_i^2)^2 .$$

Show that the Feynman propagator for the free field theory (i.e. $\lambda=0$) is of the form

$$\langle 0|T\phi_i(x)\phi_j(y)|0\rangle = \delta_{ij}\Delta_F(x-y)$$

where $\Delta_F(x-y)$ is the usual scalar propagator. Write down the Feynman rules of the theory. Compute the amplitude for the scattering $\phi_i\phi_j \to \phi_k\phi_l$ to lowest nontrivial order in λ.

9. Consider the theory of a complex scalar ψ and a real scalar ϕ with Lagrangian

$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - m^2\psi^*\psi - \frac{1}{2}M^2\phi^2 - g\psi^*\psi\phi - h|\psi|^4 - k\phi^3 - l\partial_\mu\psi\partial^\mu\psi^*\phi.$$

Draw and write down momentum-space Feynman space Feynman rules for the propagators and interactions of this theory. What are the mass dimensions of g, h, k, l? From these, identify a property of the theory.

10. Consider the scalar Yukawa theory given by the Lagrangian

$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - m^2\psi^*\psi - \frac{1}{2}M^2\phi^2 - g\psi^*\psi\phi.$$

(a) In meson decay $\phi \to \psi\bar{\psi}$, assuming that $M > 2m$, show that to lowest order in g^2 that the decay width is

$$\Gamma = \frac{g^2}{16\pi M}\sqrt{1-4m^2/M^2}.$$

Does this make sense, dimensionally?
(b) Compute the amplitude for nucleon meson scattering $\phi(p_1)+\psi(p_2) \to \phi(p_1')+\psi(p_2')$ at order g^2. Then show that

$$\frac{d\sigma}{dt} = \frac{g^4}{16\pi(s-m^2)^2(s^2+m^4+M^4-2sm^2-2sM^2-2m^2M^2)},$$

where s and t are the usual Mandelstam variables. Check that this makes sense dimensionally.