3

(i) Write down the defining relation of the Clifford algebra of γ^μ in 1+3 dimensions.

(ii) Write down the Dirac equation for a positive-frequency solution momentum space Dirac spinor $u(s, p^\mu)$ of spin s and 4-momentum p^μ.

(iii) Write down the Lagrangian density for an electron e^- of mass m and charge e coupled to an electromagnetic field A_μ.

(iv) Consider Compton scattering $\gamma(q, \epsilon_{in}(\lambda))e^-(p, s) \to \gamma(q', \epsilon_{out}(\lambda'))e^-(p', s')$ in Lorentz gauge.

(a) Write down/draw the Feynman rules needed for the calculation of the amplitude.

(b) Draw Feynman diagram(s) representing the total amplitude.

(c) Use the Feynman rules and diagrams to derive the total amplitude for Compton scattering.

(d) Calculate the modulus squared of the spin/polarisation averaged/summed amplitude in the limit $m \to 0$ in terms of the Mandelstam variables $s = (p+q)^2$ and $u = (p-q')^2$. Derive any properties of γ^μ you use from the Clifford algebra. You may assume that

$$\sum_s u(s,p)\bar{u}(s,p) = (\not{p} + m), \qquad \sum_\lambda \epsilon^*_\mu(\lambda)\epsilon_\nu(\lambda) = -\eta_{\mu\nu}.$$

for a photon polarisation $\epsilon(\lambda)$.

Marks **Comments**

(i)

$$\{\gamma^\mu,\ \gamma^\nu\} = 2\eta^{\mu\nu} I$$

[1]

(ii)

$$(\not{p} - m)u(s, p^\mu) = 0$$

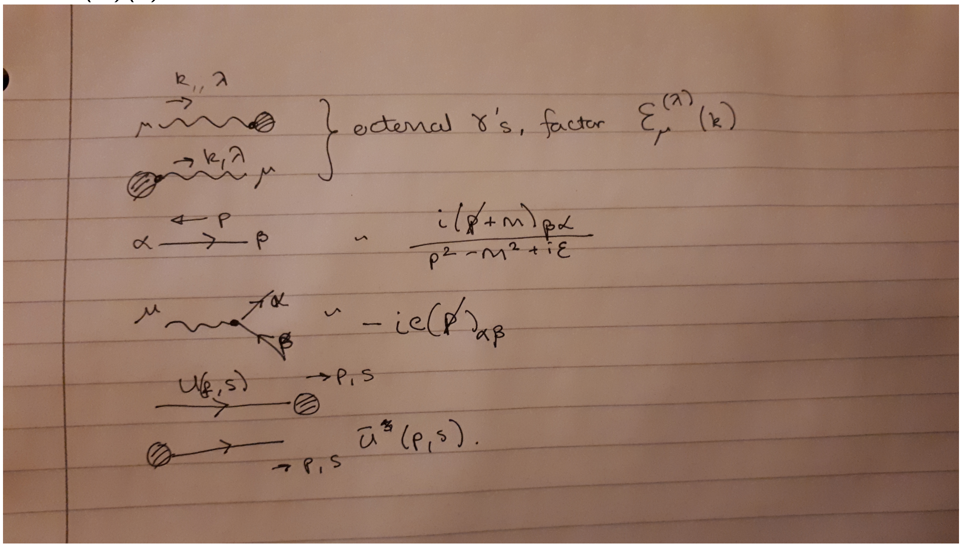
[1]

(iii)

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\not{A}\psi,$$

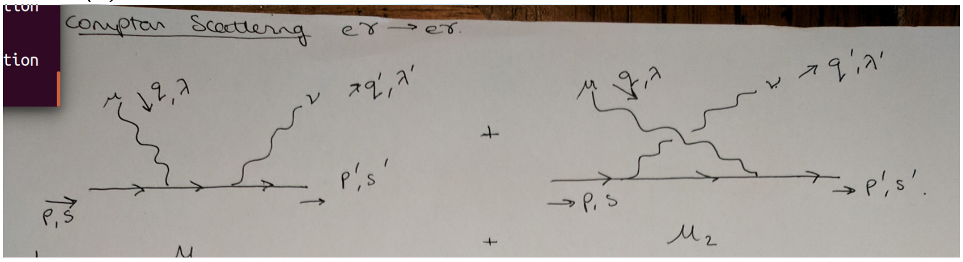
[2] *where* $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\mu$.

(iv)(a)



[3]

(b)



[2]

(c)

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = (-ie)^2 \bar{u}(s',p') {\epsilon_{out}}^*_\nu(\lambda')\gamma^\nu \frac{i(\not p + \not q)}{(p+q)^2} {\epsilon_{in}}_\mu(\lambda)\gamma^\mu u(s,p) + \begin{bmatrix} q \leftrightarrow q' \\ \epsilon_{in} \leftrightarrow \epsilon^*_{out} \end{bmatrix}$$

[1]

(d)

$$\begin{aligned} |\bar{\mathcal{M}}_1|^2 &= \frac{e^4}{4s^2} \sum_{\lambda,\lambda',s,s'} u(s',p')\bar{u}(s',p')\not\epsilon^*_{out}(\lambda')(\not p + \not q)\not\epsilon_{in}(\lambda)u(s,p)\bar{u}(s,p)\not\epsilon^*_{in}(\lambda)_\mu(\not p + \not q)\not\epsilon_{out}(\lambda') \\ &= \frac{e^4}{4s^2}\mathrm{Tr}[\not p'\gamma_\nu(\not p + \not q)\gamma_\mu \not p \gamma^\mu(\not p + \not q)\gamma^\nu] \end{aligned}$$

Now, we need

$$\gamma_\mu \not a \gamma^\mu = a^\rho \gamma_\mu(-\gamma^\mu\gamma_\rho + 2\delta^\mu_\rho) = a^\rho(-4\gamma_\rho + 2\gamma_\rho) = -2\not a.$$

$$\gamma_\mu\gamma^\nu\gamma^\rho\gamma^\mu = \gamma_\mu\gamma^\nu(-\gamma^\mu\gamma^\rho + 2\eta^{\rho\mu}) = 2\gamma^\nu\gamma^\rho + 2\gamma^\rho\gamma^\nu = 2\{\gamma^\nu,\gamma^\rho\} = 4\eta^{\nu\rho}.$$

$$\begin{aligned}\gamma_\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu &= \gamma_\mu\gamma^\nu\gamma^\rho(-\gamma^\mu\gamma^\sigma+2\eta^{\mu\sigma}) = -4\eta^{\nu\rho}\gamma^\sigma+2\gamma^\sigma\gamma^\nu\gamma^\rho \\ &= -4\eta^{\nu\rho}\gamma^\sigma+2\gamma^\sigma(-\gamma^\rho\gamma^\nu+2\eta^{\nu\rho}) = -2\gamma^\sigma\gamma^\rho\gamma^\nu.\end{aligned}$$

$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\{\gamma^\mu,\gamma^\nu\})/2 = 4\eta^{\mu\nu}$. *Thus*

$$\begin{aligned}\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 2\eta^{\mu\nu}\mathrm{Tr}(\gamma^\rho\gamma^\sigma)-\mathrm{Tr}(\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma) = 8\eta^{\mu\nu}\eta^{\rho\sigma}-2\eta^{\mu\rho}\mathrm{Tr}(\gamma^\nu\gamma^\sigma)+\mathrm{Tr}(\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma) \\ &= 8\eta^{\mu\nu}\eta^{\rho\sigma}-8\eta^{\mu\rho}\eta^{\nu\sigma}+2\eta^{\mu\sigma}\mathrm{Tr}(\gamma^\nu\gamma^\rho)-\mathrm{Tr}(\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu)\end{aligned}$$

Hence (by cyclicity of trace)

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\eta^{\mu\nu}\eta^{\rho\sigma}-4\eta^{\mu\rho}\eta^{\nu\sigma}+4\eta^{\mu\sigma}\eta^{\nu\rho}.$$

Also, using cyclicity of the trace,

$$\begin{aligned}|\bar{\mathcal{M}}_1|^2 &= \frac{e^4}{s^2}\mathrm{Tr}[\not p'(\not p+\not q)\not p(\not p+\not q)]], \qquad \not p\not p = p^2=0, \Rightarrow |\bar{\mathcal{M}}_1|^2 = \frac{e^4}{s^2}\mathrm{Tr}[\not p'\not q\not p\not q] \\ &= \frac{4e^4}{s^2}(2p'\cdot q\ p\cdot q) = -2e^4 s/u.\end{aligned}$$

For $|\bar{\mathcal{M}}_2|^2$, $s=(p+q)^2 \leftrightarrow (p-q')^2 = u$ *so* $|\bar{\mathcal{M}}_2|^2 = -2e^4 s/u$.

$$\begin{aligned}\mathcal{M}_1\mathcal{M}_2^\dagger &= \frac{e^4}{2su}\mathrm{Tr}[\not p\gamma^\mu(\not p+\not q)\gamma_\nu\not p'\gamma_\mu(\not p-\not q')\gamma^\nu] \\ &= -\frac{e^4}{2su}\mathrm{Tr}[\gamma_\mu\not p'\not p\gamma^\mu(\not p+\not q)(\not p-\not q')] \\ &= -\frac{2e^4}{su}p\cdot p'\mathrm{Tr}[(\not p+\not q)(\not p-\not q')]\end{aligned}$$

The trace is $4(p^2-p\cdot q'+q\cdot p-q\cdot q')$ *but* $p=p'-q'-q$ *so trace is* $4(q\cdot(p'+q'-q)-p\cdot q'-q\cdot q')=0$ *in the massless limit since* $q\cdot p' = p\cdot q'$. *Hence*

$$|\bar{\mathcal{M}}|^2 = -2e^4\left(\frac{u}{s}+\frac{s}{u}\right).$$

[10]

[20]
