Quantum Field Theory: Example Sheet 1

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Question 1

Suppose that $\phi(x)$ is a solution to the Klein–Gordon equation,

$$\left[\eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}+m^2\right]\phi(x)=0. \tag{1.1}$$

Consider an active transformation that boosts the field according to the Lorentz transformation Λ,[1]

$$\phi(x)\xrightarrow{\Lambda}\phi'(x)=\phi(\Lambda^{-1}x)\equiv\phi(y), \tag{1.2}$$

where in the last equality, we write $y^\mu=(\Lambda^{-1})^\mu{}_\nu x^\nu$. To prove that $\phi(y)\equiv\phi(\Lambda^{-1}x)$ is also a solution to the Klein–Gordon equation, we can act on it with the same operator in square brackets. The chain rule tells us that

$$\frac{\partial}{\partial x^\nu}=\frac{\partial y^\mu}{\partial x^\nu}\frac{\partial}{\partial y^\mu}=(\Lambda^{-1})^\mu{}_\nu\frac{\partial}{\partial y^\mu}, \tag{1.3}$$

thus

$$\begin{aligned}\left[\eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}+m^2\right]\phi(y)&=\left[\eta^{\mu\nu}(\Lambda^{-1})^\alpha{}_\mu(\Lambda^{-1})^\beta{}_\nu\frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial y^\beta}+m^2\right]\phi(y)\\&=\left[\eta^{\alpha\beta}\frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial y^\beta}+m^2\right]\phi(y).\end{aligned} \tag{1.4}$$

We arrive at the second line after using the fact that any Lorentz transformation Λ (and its inverse) must satisfy the condition

$$\eta^{\mu\nu}(\Lambda^{-1})^\alpha{}_\mu(\Lambda^{-1})^\beta{}_\nu=\eta^{\alpha\beta}. \tag{1.5}$$

The second line in Eq. (1.4) is exactly the LHS of Eq. (1.1) up to renaming $x\to y$, thus $\phi(y)\equiv\phi(\Lambda^{-1}x)$ is indeed a solution to the Klein–Gordon equation.

Question 2

Consider the complex field $\psi(x)$ governed by the Lagrangian (density)

$$\mathcal{L}=\partial_\mu\psi^*\partial^\mu\psi-m^2\psi^*\psi-\frac{\lambda}{2}(\psi^*\psi)^2. \tag{2.1}$$

We determine its equations of motion by evaluating the Euler–Lagrange equation

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)}\right)=\frac{\partial\mathcal{L}}{\partial\psi^*}, \tag{2.2}$$

which yields

$$\partial_\mu\partial^\mu\psi+m^2\psi+\lambda(\psi^*\psi)\psi=0. \tag{2.3}$$

We also obtain a similar equation for ψ^* after interchanging $\psi\leftrightarrow\psi^*$ in the above two steps.

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[1] See Sec. 1.2 of David Tong's lecture notes [1] if you're unsure about why the inverse transformation Λ^{-1} appears in the argument of the field.

Proposition 2.1: *The Lagrangian is invariant under the infinitesimal transformation*

$$\delta\psi = i\alpha\psi, \quad \delta\psi^* = -i\alpha\psi^*. \tag{2.4}$$

Proof.—Replacing $\psi \to \psi + \delta\psi$ and $\psi^* \to \psi^* + \delta\psi^*$ in the Lagrangian, the terms linear in the constant α are

$$\delta\mathcal{L} = \partial_\mu \delta\psi^* \partial^\mu \psi + \partial_\mu \psi^* \partial^\mu \delta\psi - m^2(\delta\psi^*\psi + \psi^*\delta\psi) - \lambda(\psi^*\psi)(\delta\psi^*\psi + \psi^*\delta\psi). \tag{2.5}$$

Using the explicit expressions for $\delta\psi$ and $\delta\psi^*$ given in Eq. (2.4), we find that the first two terms in $\delta\mathcal{L}$ cancel each other, the next two terms proportional to m^2 cancel each other, and the last two terms proportional to λ also cancel each other. Since $\delta\mathcal{L} = 0$, this transformation leaves the Lagrangian invariant.

Note that the same conclusion can be obtained by instead considering the global U(1) transformation $\psi \to e^{i\alpha}\psi$ and $\psi^* \to e^{-i\alpha}\psi^*$, which would recover Eq. (2.4) in the infinitesimal limit $\alpha \ll 1$. It is easy to see that the product $\psi^*\psi$ is unchanged under this transformation, and since α is a constant, the same is true of the derivative terms $\partial_\mu\psi^*\partial^\mu\psi$. ■

Proposition 2.2: *The Noether current associated with this symmetry* (2.4) *is*

$$j^\mu = i\left(\psi\partial^\mu\psi^* - \psi^*\partial^\mu\psi\right). \tag{2.6}$$

Proof.—A continuous transformation is a symmetry of the theory if it leaves the Lagrangian invariant up to a total derivative term, $\delta\mathcal{L} = \partial_\mu F^\mu$. Noether's theorem[2] states that any such symmetry has an associated conserved current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)} \cdot \delta\Psi - F^\mu, \tag{2.7}$$

where Ψ is a placeholder for all the fields in the theory. In our case, $\Psi = (\psi, \psi^*)$.

The symmetry in Eq. (2.4) is an example of an *internal* symmetry, which is a transformation of the form $\Psi \to G\Psi$, where G is a group element. Internal symmetries always yield $F^\mu = 0$ (as we saw explicitly when proving Proposition 2.1).[3] Thus, the Noether current is

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)}\delta\psi^* = i\alpha(\psi\partial^\mu\psi^* - \psi^*\partial^\mu\psi). \tag{2.8}$$

Since this current is conserved regardless of the value of the constant α, we can discard it to recover the desired result. ■

We can verify explicitly that this current is conserved by evaluating $\partial_\mu j^\mu$ and showing that it vanishes. Specifically,

$$\begin{aligned} -i\partial_\mu j^\mu &= \partial_\mu(\psi\partial^\mu\psi^* - \psi^*\partial^\mu\psi) \\ &= \psi(\partial_\mu\partial^\mu\psi^*) - \psi^*(\partial_\mu\partial^\mu\psi) \\ &= \psi\left[-m^2\psi^* - \lambda(\psi^*\psi)\psi^*\right] - \psi^*\left[-m^2\psi - \lambda(\psi^*\psi)\psi\right] = 0. \end{aligned} \tag{2.9}$$

The last line follows from using the equation of motion (2.3) and an identical one with $\psi \leftrightarrow \psi^*$.

[2] See, e.g., Sec. 1.3.1 of David Tong's lecture notes [1], or p. 17–18 of Peskin and Schroeder [2].

[3] *External* symmetries involving transformations on the underlying spacetime (e.g., Lorentz transformations $x \to \Lambda x$) will generically give a nonzero F^μ.

Question 3

Consider a triplet of real fields ϕ_a, with $a \in \{1, 2, 3\}$, governed by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2}m^2 \phi_a \phi_a. \tag{3.1}$$

Proposition 3.1: *The Noether currents associated with the SO(3) symmetry $\phi_a \to \phi_a + \theta \epsilon_{abc}\eta_b \phi_c$, where η_a is an arbitrary constant unit vector, are*

$$(j_a)^\mu = \epsilon_{abc}\phi_b \partial^\mu \phi_c. \tag{3.2}$$

Proof.—This transformation is an internal symmetry of the theory, so we have $F^\mu = 0$ just like in Question 2.[4] Accordingly, the Noether current is

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a = (\partial^\mu \phi_a)\theta\epsilon_{abc}\eta_b\phi_c = \theta\epsilon_{abc}\eta_a\phi_b\partial^\mu\phi_c, \tag{3.3}$$

having used the antisymmetry of ϵ_{abc} and the freedom to relabel indices to obtain the final expression. This is conserved regardless of the value of the constant θ, so we can discard it in our definition of the current j^μ. Additionally, this current is also conserved for any choice of unit vector η_a, so we should "strip it out" to obtain three independently conserved currents—one for each independent direction in the field space $\mathbb{R}^3$.

The systematic way to do this is to decompose $\eta_a = N_b(e^b)_a$, where N_b are just numbers, and $(e^b)_a$ forms a basis that spans $\mathbb{R}^3$. Note that the inner index b inside the brackets is labelling the different basis vectors, while the outer index a is the vector index. This allows us to decompose the current as $j^\mu = N_a(j^a)^\mu$, where

$$(j^a)^\mu = (e^a)_d \epsilon_{dbc}\phi_b\partial^\mu\phi_c. \tag{3.4}$$

We recover the desired result by choosing the basis vectors to be

$$(e^1)_d = (1, 0, 0), \quad (e^2)_d = (0, 1, 0), \quad (e^3)_d = (0, 0, 1); \tag{3.5}$$

which can be written succintly as $(e^a)_d = \delta^a_d$. Note that the indices a, b can be summed over regardless of whether they are raised or lowered since the metric on $\mathbb{R}^3$ is just diag$(1, 1, 1)$. ■

By definition, each conserved current has an associated charge

$$Q_a = \int \mathrm{d}^3x\, (j_a)^0 = \int \mathrm{d}^3x\, \epsilon_{abc}\phi_b\dot{\phi}_c. \tag{3.6}$$

Note that this result differs by a minus sign from the expected answer given in the question. This is not a problem, since we are always free to define currents and their associated charges up to a constant.

Proposition 3.2: *The charges Q_a are conserved.*

Proof.—We can verify this explicitly by evaluating its derivative,

$$\frac{\mathrm{d}Q_a}{\mathrm{d}t} = \int \mathrm{d}^3x\, \frac{\mathrm{d}}{\mathrm{d}t}(\epsilon_{abc}\phi_b\dot{\phi}_c) = \int \mathrm{d}^3x\, \epsilon_{abc}\phi_b\ddot{\phi}_c. \tag{3.7}$$

When using the product rule to obtain the last equality, note that we also get a term $\epsilon_{abc}\dot{\phi}_b\dot{\phi}_c$, but this vanishes from the antisymmetry of ϵ_{abc}. We then proceed by using the fact that each component

[4] If we were to do this explicitly, we would find that terms in $\delta\mathcal{L}$ are either proportional to $\epsilon_{abc}\phi_b\phi_c$ or $\epsilon_{abc}\partial_\mu\phi_b\partial^\mu\phi_c$, which vanish due to the antisymmetry of ϵ_{abc}.

$\phi_a(x)$ is a solution to Klein–Gordon equation,

$$(\partial_\mu \partial^\mu + m^2)\phi_a = \ddot{\phi}_a - \nabla^2 \phi_a + m^2 \phi_a = 0. \tag{3.8}$$

This allows us to write

$$\frac{\mathrm{d}Q_a}{\mathrm{d}t} = \int \mathrm{d}^3x \; \epsilon_{abc} \phi_b (\nabla^2 \phi_c - m^2 \phi_c) = -\int \mathrm{d}^3x \; \epsilon_{abc} \nabla \phi_b \cdot \nabla \phi_c. \tag{3.9}$$

In obtaining the last equality, we again use the antisymmetry of ϵ_{abc} to deduce that $m^2 \epsilon_{abc} \phi_b \phi_c = 0$, and we integrate by parts to replace $\phi_b \nabla^2 \phi_c \to -\nabla \phi_b \cdot \nabla \phi_c$. In this step, we are making the reasonable assumption that the fields decay sufficiently fast at large distances such that the boundary term at spatial infinity can be neglected. Finally, the antisymmetry of ϵ_{abc} is used once more to conclude that $\mathrm{d}Q_a / \mathrm{d}t = 0$, as expected. ■

Question 4

Proposition 4.1: *A Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ is a transformation that preserves the invariant interval $\eta_{\mu\nu} x^\mu x^\nu = \eta_{\mu\nu} x'^\mu x'^\nu$. This can be used to deduce that*

$$\eta_{\mu\nu} = \eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu. \tag{4.1}$$

Proof.—By writing $x'^\alpha = \Lambda^\alpha{}_\mu x^\mu$, we have that

$$\eta_{\alpha\beta} x'^\alpha x'^\beta = \eta_{\alpha\beta} (\Lambda^\alpha{}_\mu x^\mu)(\Lambda^\beta{}_\nu x^\nu) = (\eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu) x^\mu x^\nu. \tag{4.2}$$

As we require this to be equivalent to $\eta_{\mu\nu} x^\mu x^\nu$ for arbitrary x^μ, it must be that Eq. (4.1) is true.[5] ■

Proposition 4.2: *If $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \alpha \omega^\mu{}_\nu$ is an infinitesimal Lorentz transformation[6] with $\alpha \ll 1$, then $\omega_{\mu\nu}$ is antisymmetric.*

Proof.—When expanded to linear order in α, we have that

$$\eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \simeq \eta_{\alpha\beta} \left(\delta^\alpha_\mu \delta^\beta_\nu + \alpha \omega^\alpha{}_\mu \delta^\beta_\nu + \alpha \delta^\alpha_\mu \omega^\beta{}_\nu \right) = \eta_{\mu\nu} + \alpha(\omega_{\nu\mu} - \omega_{\mu\nu}). \tag{4.3}$$

This is equal to $\eta_{\mu\nu}$ only if $\omega_{\mu\nu} = -\omega_{\nu\mu}$. ■

Let us consider examples of the matrix $\omega^\mu{}_\nu$. A rotation through an angle θ about the x^3-axis corresponds to the Lorentz transformation

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \simeq \delta^\mu_\nu + \theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.4}$$

The latter expression holds for small angles $\theta \ll 1$, which is serving as the small parameter α used earlier.

[5] More precisely, only the symmetric part of Eq. (4.1) is true. But, by construction, the Minkowski metric and the product $\eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu$ are manifestly symmetric under the interchange $\mu \leftrightarrow \nu$, so the proof is complete.

[6] A good way to think about this is to consider a one-parameter family of transformations defined by the differentiable map $\Lambda : \mathcal{I} \to \mathrm{SO}(3,1)^+$, where the interval $\mathcal{I} \subset \mathbb{R}$ is parametrized by α, and $\mathrm{SO}(3,1)^+$ is the part of the Lorentz group smoothly connected to the identity. Without loss of generality, our map can be defined such that $\Lambda^\mu{}_\nu(0) = \delta^\mu_\nu$. Then, an infinitesimal Lorentz transformation is obtained by Taylor expanding about the origin to get $\Lambda^\mu{}_\nu(\alpha) = \Lambda^\mu{}_\nu(0) + \alpha \dot{\Lambda}^\mu{}_\nu(0) + \mathcal{O}(\alpha^2)$. The first derivative $\dot{\Lambda}^\mu{}_\nu(0)$ is what we call $\omega^\mu{}_\nu$.

For a Lorentz boost along the x^1-axis, we instead have

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \simeq \delta^\mu_\nu + v \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.5}$$

where $\gamma = 1/\sqrt{1-v^2}$, and the latter expression is valid for small velocities $v \ll 1$. Notice that $\omega^\mu{}_\nu$ need not be antisymmetric; only $\omega_{\mu\nu}$ or $\omega^{\mu\nu}$ should be antisymmetric.

Question 5

Consider the infinitesimal Lorentz transformation $x^\mu \to \Lambda(\alpha)^\mu{}_\nu x^\nu$ with $\Lambda(\alpha)^\mu{}_\nu = \delta^\mu_\nu + \alpha\omega^\mu{}_\nu$, under which a scalar field $\phi(x)$ transforms according to Eq. (1.2). To get an expression for the inverse $\Lambda^{-1}(\alpha)$, notice that applying the transformation $\Lambda(-\alpha) \circ \Lambda(\alpha)$ is the same as not performing a transformation at all, so it must be that $\Lambda(-\alpha) \equiv \Lambda^{-1}(\alpha)$.[7] Hence,

$$\phi(x) \xrightarrow{\Lambda(\alpha)} \phi\big(\Lambda(-\alpha)x\big) = \phi(x) - \alpha\omega^\mu{}_\nu x^\nu \partial_\mu \phi(x). \tag{5.1}$$

Proposition 5.1: *Under the infinitesimal Lorentz transformation $\Lambda(\alpha)$, the Lagrangian $\mathcal{L} \equiv \mathcal{L}(\phi, \partial\phi)$ is unchanged up to the total derivative*

$$\delta\mathcal{L} = -\alpha\partial_\mu(\omega^\mu_\nu x^\nu \mathcal{L}). \tag{5.2}$$

Proof.—The Lagrangian $\mathcal{L}$ is a function of $\phi(x)$ and its derivative $\partial_\mu\phi(x)$, but these objects are themselves functions of x, so we can also think of $\mathcal{L}$ as just being a function of x. This is a scalar quantity, so—analogous to Eq. (5.1)—must transform as

$$\mathcal{L}(x) \xrightarrow{\Lambda(\alpha)} \mathcal{L}\big(\Lambda(-\alpha)x\big) = \mathcal{L}(x) - \alpha\omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L}(x). \tag{5.3}$$

We read off $\delta\mathcal{L}$ from the second term, but this is not yet in the desired form. We proceed by using the product rule to write

$$\delta\mathcal{L} = -\alpha\omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = -\alpha\partial_\mu(\omega^\mu{}_\nu x^\nu \mathcal{L}) + \alpha\mathcal{L}\partial_\mu(\omega^\mu{}_\nu x^\nu). \tag{5.4}$$

The last term is proportional to $\partial_\mu(\omega^\mu{}_\nu x^\nu) = \omega^\mu{}_\nu \delta^\nu_\mu = \omega^\mu{}_\mu$, which vanishes since the constant tensor $\omega_{\mu\nu}$ is antisymmetric. This yields the desired result. ■

Proposition 5.2: *A Lorentz-invariant theory admits the conserved current*

$$j^\mu = -\omega^\rho{}_\nu (T^\mu{}_\rho x^\nu). \tag{5.5}$$

Proof.—Since $\Lambda^\mu{}_\nu(\alpha)$ is a symmetry of the theory, we use Noether's theorem to deduce that the current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - F^\mu = -\alpha\omega^\rho{}_\nu x^\nu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\rho\phi - \delta^\mu_\rho\mathcal{L}\right] \tag{5.6}$$

is conserved, having read off $\delta\phi = -\alpha\omega^\rho{}_\nu x^\nu \partial_\rho\phi$ and $F^\mu = -\alpha\omega^\mu{}_\nu x^\nu \mathcal{L}$ from Eqs. (5.1) and (5.2), respectively. As always, the constant α can be discarded, and we note that the object in square brackets is exactly the energy–momentum tensor $T^\mu{}_\rho$ by definition. ■

[7] Alternatively, we can also arrive at this conclusion by multiplying Eq. (4.1) by $(\Lambda^{-1})^\nu{}_\sigma$, using the identity $\Lambda^\beta{}_\nu(\Lambda^{-1})^\nu{}_\sigma = \delta^\beta_\sigma$, and then judiciously manipulating indices.

Proposition 5.3: *The current in Eq. (5.5) leads to six independently conserved charges,*

$$Q^{\alpha\beta} = \int \mathrm{d}^3x \left(x^\alpha T^{0\beta} - x^\beta T^{0\alpha}\right), \tag{5.7}$$

where $Q^{\alpha\beta} = -Q^{\beta\alpha}$ is an antisymmetric tensor.

Proof.—The current in Eq. (5.5) is conserved for any constant antisymmetric tensor $\omega_{\mu\nu}$, hence we should strip it out to obtain six independently conserved currents. In analogy with Question 3, we might think to do this by decomposing $\omega_{\mu\nu} = \Omega_A(\mathcal{M}^A)_{\mu\nu}$, where Ω_A is just a set of numbers, and $(\mathcal{M}^A)_{\mu\nu}$ forms a basis for the set of 4×4 antisymmetric matrices, labelled by the index $A \in \{1, 2, \dots, 6\}$. However, it turns out to be difficult to work with the index A. Instead, it is more convenient to label the six different basis elements by an antisymmetric pair of indices $\alpha, \beta \in \{0, 1, 2, 3\}$, such that

$$\omega_{\mu\nu} = \frac{1}{2}\Omega_{\alpha\beta}(\mathcal{M}^{\alpha\beta})_{\mu\nu}. \tag{5.8}$$

Substituting this decomposition into Eq. (5.5) allows us to write $j^\mu = \frac{1}{2}\Omega_{\alpha\beta}(j^{\alpha\beta})^\mu$, from which we can read off

$$(j^{\alpha\beta})^\mu = -(\mathcal{M}^{\alpha\beta})_{\rho\nu}T^{\mu\rho}x^\nu. \tag{5.9}$$

The most natural choice for the basis elements $(\mathcal{M}^{\alpha\beta})_{\rho\nu}$ is

$$\begin{aligned}
(\mathcal{M}^{01})_{\mu\nu} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\mathcal{M}^{02})_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\mathcal{M}^{03})_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\
(\mathcal{M}^{12})_{\mu\nu} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\mathcal{M}^{13})_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad (\mathcal{M}^{23})_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix};
\end{aligned} \tag{5.10}$$

which can be written succinctly in index notation as $(\mathcal{M}^{\alpha\beta})_{\rho\nu} = \delta^\alpha_\rho\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\rho$. Substituting this back into Eq. (5.9) yields

$$(j^{\alpha\beta})^\mu = x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha}. \tag{5.11}$$

We use this to obtain the six independent charges

$$Q^{\alpha\beta} = \int \mathrm{d}^3x\, (j^{\alpha\beta})^0, \tag{5.12}$$

which yield the desired result. ■

Proposition 5.4: *Rotational invariance leads to three conserved charges*

$$Q_i = \frac{1}{2}\epsilon_{ijk}\int \mathrm{d}^3x \left(x^j T^{0k} - x^k T^{0j}\right), \tag{5.13}$$

which are interpreted as measuring the total angular momentum of the field about the x^i-axes.

Proof.—The generators of spatial rotations correspond to the elements in the second row of Eq. (5.10), which can be selected by focussing on the spatial indices $(\alpha, \beta) \to (i, j)$ in Eq. (5.7). This gives us three charges

$$Q^{ij} = \int \mathrm{d}^3x \left(x^i T^{0j} - x^j T^{0i}\right). \tag{5.14}$$

Rather than count with the antisymmetric pair of indices (i, j), we can label these charges by the single index $i \in \{1, 2, 3\}$ by defining

$$Q_i = \frac{1}{2}\epsilon_{ijk}Q^{jk}. \tag{5.15}$$

Substituting Eq. (5.14) into this definition returns the desired result. ■

The remaining three charges associated with Lorentz boosts are

$$Q^{0i} = \int \mathrm{d}^3x \left(x^0 T^{0i} - x^i T^{00}\right). \tag{5.16}$$

Their physical significance can be made more transparent by taking their time derivative. Since they must be conserved, it follows that

$$0 = \frac{\mathrm{d}Q^{0i}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int \mathrm{d}^3x\; x^0 T^{0i}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\int \mathrm{d}^3x\; x^i T^{00}\right), \tag{5.17}$$

which can be rewritten as

$$\frac{\mathrm{d}X^i}{\mathrm{d}t} = P^i + t\frac{\mathrm{d}P^i}{\mathrm{d}t}. \tag{5.18}$$

The RHS is expressed in terms of the total linear momentum of the field (T^{0i} is the momentum density),

$$P^i = \int \mathrm{d}^3x\; T^{0i}, \tag{5.19}$$

which is conserved since the theory is also invariant under spacetime translations. Consequently, the RHS of Eq. (5.18) is a constant. Since

$$X^i = \int \mathrm{d}^3x\; x^i T^{00} \tag{5.20}$$

is the centre of energy of the field (T^{00} is the energy density), we learn that symmetry under Lorentz boosts guarantees that the centre of energy of the field travels with constant velocity.

Question 6

In pure vacuum, electromagnetic fields are governed by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{6.1}$$

where the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is defined in terms of the gauge field A_μ.

Proposition 6.1: *The Lagrangian is invariant under the gauge transformation*

$$A_\mu \to A_\mu + \partial_\mu \xi. \tag{6.2}$$

Proof.—The field strength tensor transforms as

$$\begin{aligned} F_{\mu\nu} \to F'_{\mu\nu} &= \partial_\mu(A_\nu + \partial_\nu\xi) - \partial_\nu(A_\mu + \partial_\mu\xi) \\ &= (\partial_\mu A_\nu - \partial_\nu A_\mu) + (\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\xi. \end{aligned} \tag{6.3}$$

In the second line, the first term is just the original field strength tensor $F_{\mu\nu}$, and the second term vanishes since partial derivatives commute. Thus, $F_{\mu\nu}$ is invariant under a gauge transformation, and since $\mathcal{L}$ is built up only from contractions of $F_{\mu\nu}$, it follows that $\mathcal{L}$ is also gauge invariant. ■

Proposition 6.2: *Translational invariance $x^\mu \to x^\mu - \epsilon a^\mu$ leads to the conserved "canonical" energy–*

momentum tensor

$$T^{\mu\nu} = F^{\rho\mu}\partial^\nu A_\rho + \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}. \tag{6.4}$$

Proof.—Under a translation, the gauge field transforms as

$$A_\mu(x) \to A_\mu(x+\epsilon a) = A_\mu(x) + \epsilon a^\nu \partial_\nu A_\mu(x) + \mathcal{O}(\epsilon^2), \tag{6.5}$$

and similarly the Lagrangian picks up the total derivative

$$\mathcal{L}(x) \to \mathcal{L}(x+\epsilon a) = \mathcal{L}(x) + \epsilon a^\nu \partial_\nu \mathcal{L}(x) + \mathcal{O}(\epsilon)^2. \tag{6.6}$$

From these two equations, we can read off the infinitesimal changes $\delta A_\mu = a^\nu \partial_\nu A_\mu$ and $F^\mu = a^\mu \mathcal{L}$, having already dropped the overall factor of ϵ that is not needed. Noether's theorem gives us the conserved current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\delta A_\nu - F^\mu = -F^{\mu\nu}(a^\rho \partial_\rho A_\nu) - a^\mu\left(-\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\right). \tag{6.7}$$

In obtaining the first term, the partial derivative can be evaluated by using the fact that

$$\frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)} = \frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial_\rho A_\sigma - \partial_\sigma A_\rho) = \delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho. \tag{6.8}$$

We simplify Eq. (6.7) to obtain

$$j^\mu = a_\nu\left(F^{\rho\mu}\partial^\nu A_\rho + \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{6.9}$$

where the antisymmetry of the field strength tensor $F^{\mu\nu} = -F^{\nu\mu}$ has been used to get rid of the minus sign in the first term. As always, there are actually four conserved currents here, since there are four independent ways of choosing the vector a^μ. Stripping it out leads to the desired result for the energy–momentum tensor $T^{\mu\nu}$.[8] ■

Proposition 6.3: *$T^{\mu\nu}$ is neither symmetric nor gauge invariant.*

Proof.—To show that it is not symmetric, simply consider the quantity

$$T^{\mu\nu} - T^{\nu\mu} = F^{\rho\mu}\partial^\nu A_\rho - F^{\rho\nu}\partial^\mu A_\rho, \tag{6.10}$$

and notice that it does not vanish. The tensor $T^{\mu\nu}$ is also manifestly not gauge invariant, since it depends explicitly on the gauge field A_μ. Under the transformation in Eq. (6.2), we obtain

$$T^{\mu\nu} \to T^{\mu\nu} + F^{\rho\mu}\partial^\nu\partial_\rho\xi, \tag{6.11}$$

where the second term is nonvanishing. ■

We overcome these problematic issues with $T^{\mu\nu}$ by defining the "physical" energy–momentum tensor[9]

$$\Theta^{\mu\nu} = T^{\mu\nu} - F^{\rho\mu}\partial_\rho A^\nu. \tag{6.12}$$

Proposition 6.4: *$\Theta^{\mu\nu}$ is symmetric, gauge invariant, and traceless.*

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[8] We could have done this meticulously, as we did in Questions 3 and 5, by decomposing $a_\mu = \alpha_\nu (e^\nu)_\mu$, where α_ν are just numbers and $(e^\nu)_\mu$ form a basis for Minkowski space. Then the current can be written as $j^\mu = \alpha_\nu (j^\nu)^\mu$. Choosing the basis $(e^\nu)_\mu = \delta^\nu_\mu$ returns the desired result, $T^{\mu\nu} = (j^\nu)^\mu$. Once you get the hang of it, however, stripping out constant vectors or tensors from the Noether current becomes an automatic process.

[9] Also known as the improved energy–momentum tensor or the Belinfante–Rosenfeld energy–momentum tensor.

Proof.—It is straightforward to show that Eq. (6.12) can be simplified to read

$$\Theta^{\mu\nu} = F^{\rho\mu}F^{\nu}{}_{\rho} + \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}. \tag{6.13}$$

This is manifestly gauge invariant since it is constructed only out of the field strength tensor. The first term can be rewritten as

$$F^{\rho\mu}F^{\nu}{}_{\rho} = -\eta_{\rho\sigma}F^{\rho\mu}F^{\sigma\nu}, \tag{6.14}$$

and the second term is proportional to $\eta^{\mu\nu}$, thus $\Theta^{\mu\nu}$ is also manifestly symmetric. Finally, we evaluate its trace to find

$$\Theta^{\mu}{}_{\mu} = F^{\rho\mu}F_{\mu\rho} + \frac{1}{4}(\eta_{\mu\nu}\eta^{\mu\nu})F_{\rho\sigma}F^{\rho\sigma} = 0, \tag{6.15}$$

having used $\eta_{\mu\nu}\eta^{\mu\nu} = \delta^{\mu}_{\mu} = 4$ for a four-dimensional spacetime and the antisymmetry $F_{\mu\rho} = -F_{\rho\mu}$ in the last step. ■

Proposition 6.5: $\Theta^{\mu\nu}$ *defines four conserved currents.*

Proof.—We prove this by evaluating $\partial_\mu\Theta^{\mu\nu}$ and showing that it vanishes. By construction, $\partial_\mu T^{\mu\nu} = 0$, thus

$$\partial_\mu\Theta^{\mu\nu} = \partial_\mu T^{\mu\nu} + \partial_\mu(F^{\rho\mu}\partial_\rho A^\nu) = F^{\rho\mu}\partial_\mu\partial_\rho A^\nu. \tag{6.16}$$

In the last step, we used the fact that the fields satisfy the equations of motion, $\partial_\mu F^{\rho\mu} = \partial_\mu F^{\mu\rho} = 0$. Since $F^{\rho\mu}$ is antisymmetric while $\partial_\mu\partial_\rho$ is symmetric, it follows that $\partial_\mu\Theta^{\mu\nu} = 0$. Note that, due to its symmetry, this automatically implies that we also have $\partial_\nu\Theta^{\mu\nu} = 0$. ■

Aside 6.1: Let us say a few more words about the different energy–momentum tensors we encountered in this question. We'll frame this discussion by first computing the conserved currents imposed by Lorentz invariance. For the infinitesimal transformation $\Lambda^{\mu}{}_{\nu}(\alpha) = \delta^{\mu}{}_{\nu} + \alpha\omega^{\mu}{}_{\nu}$, it can be shown that the six conserved currents are

$$(j^{\alpha\beta})^\mu = x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha} + (s^{\alpha\beta})^\mu, \tag{6.17}$$

where $(s^{\alpha\beta})^\mu = F^{\alpha\mu}A^\beta - F^{\beta\mu}A^\alpha$. Deriving this is a worthwhile exercise left to the reader.

Comparing this with the expression for $(j^{\alpha\beta})^\mu$ for a scalar field given in Eq. (5.11), we notice that there is this extra bit $(s^{\alpha\beta})^\mu$ which arises because the gauge field A_μ transforms as a Lorentz covector. Accordingly, we should interpret $(s^{\alpha\beta})^\mu$ as representing the contribution to this current from the intrinsic spin of the photon.

By construction, the current in Eq. (6.17) is conserved, and recall we also have $\partial_\mu T^{\mu\nu} = 0$. These imply that

$$\partial_\mu(j^{\alpha\beta})^\mu = T^{\alpha\beta} - T^{\beta\alpha} + \partial_\mu(s^{\alpha\beta})^\mu = 0. \tag{6.18}$$

Because it is not symmetric, the quantity $T^{\alpha\beta} - T^{\beta\alpha}$ does not vanish on its own. It is only the sum of the energy–momentum density contained in $T^{\alpha\beta}$ and in the "spin gradients" $\partial_\mu(s^{\alpha\beta})^\mu$ that together ensure that the Lorentz currents $(j^{\alpha\beta})^\mu$ are conserved. This motivates the definition of the physical energy–momentum tensor $\Theta^{\mu\nu}$ which should satisfy

$$\partial_\mu(j^{\alpha\beta})^\mu = \Theta^{\alpha\beta} - \Theta^{\beta\alpha} = 0 \tag{6.19}$$

to preserve Lorentz invariance. It follows that $\Theta^{\mu\nu}$ should be constructed from a linear combination of $T^{\mu\nu}$ and gradients of $(s^{\mu\nu})^\alpha$ in a way that ensures $\Theta^{\mu\nu}$ is symmetric and conserved. Belinfante and Rosenfeld showed that a valid choice is [3]

$$\Theta^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\rho \left[(s^{\mu\nu})^\rho + (s^{\nu\rho})^\mu + (s^{\mu\rho})^\nu \right]. \tag{6.20}$$

Substituting the definition of $(s^{\mu\nu})^\rho$ into this expression returns Eq. (6.12), after using the equations of motion, $\partial_\mu F^{\mu\nu} = 0$.

It is worth stressing that nothing went wrong with Noether's theorem when deriving $T^{\mu\nu}$, which are still valid conserved currents arising from translational invariance. We only run into problems if we try to *interpret* $T^{\mu\nu}$ as measuring the energy–momentum density of the field. When we do this, we are missing out a key part due to the intrinsic spin of the photon. The physical energy–momentum tensor $\Theta^{\mu\nu}$ is what we measure in experiments.

One might still find the construction of $\Theta^{\mu\nu}$ a bit ad hoc. How can we be certain that this really is the physical energy–momentum tensor of the field? For this, we turn to Einstein. General relativity tells us that all forms of energy must gravitate and curve spacetime, so the physical energy–momentum tensor of a given field is whatever appears on the RHS of the Einstein field equations,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}\Theta^{\mu\nu}. \tag{6.21}$$

Given a Lagrangian $\mathcal{L}$ for the matter content, we derive[†]

$$\Theta^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g_{\mu\nu}} = -2\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} + g_{\mu\nu}\mathcal{L}. \tag{6.22}$$

See the Part III General Relativity course for more details.

[†]Note that these last two equations are written in terms of a metric with signature $(-,+,+,+)$, which is more popular in the general relativity literature.

Question 7

Consider a massive vector field[10] C_μ governed by the action

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 C_\mu C^\mu, \tag{7.1}$$

where the field strength tensor $F_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$. It is straightforward to show by using the Euler–Lagrange equations that this field satisfies the equation of motion

$$\partial_\mu F^{\mu\nu} + m^2 C^\nu = 0. \tag{7.2}$$

Proposition 7.1: *When $m \neq 0$, the field satisfies the constraint $\partial_\mu C^\mu = 0$.*

Proof.—Take the derivative ∂_ν of Eq. (7.2) to obtain $\partial_\mu\partial_\nu F^{\mu\nu} + m^2\partial_\nu C^\nu = 0$. The first term on the LHS vanishes due to the antisymmetry of $F^{\mu\nu}$, and the second term gives us the desired result. ■

Proposition 7.2: *C_0 is a nondynamical field, which can be eliminated completely in terms of the remaining degrees of freedom C_i.*

Proof.—The constraint $\partial_\mu C^\mu = 0$ implies that $\dot{C}^0 = -\partial_i C^i$. We substitute this back into the $\nu = 0$ component of Eq. (7.2) to obtain

$$\partial_\mu F^{\mu 0} + m^2 C^0 = 0. \tag{7.3}$$

[10]In the literature, this is also often called a Proca field.

Since $F^{00} = 0$, we expand $\partial_\mu F^{\mu 0} = \partial_i F^{i0} = \partial_i(\partial^i C^0 - \partial^0 C^i)$ to obtain

$$\partial_i \partial^i C_0 + m^2 C_0 = \partial_i \dot{C}^i, \tag{7.4}$$

or equivalently,

$$(\nabla^2 - m^2) C_0 = -\partial_i \dot{C}^i, \tag{7.5}$$

where the 0 index can be lowered with the Minkowski metric, $C^0 \equiv C_0$. This is the inhomogeneous modified Helmholtz equation.[11] Its solution can be determined uniquely by the method of Green's functions,

$$C_0(t, \mathbf{x}) = \int \mathrm{d}^3 x' \, \frac{\partial_i \dot{C}^i(t, \mathbf{x}')}{4\pi |\mathbf{x} - \mathbf{x}'|} e^{-m|\mathbf{x}-\mathbf{x}'|}, \tag{7.6}$$

provided we choose boundary conditions that forbid any homogeneous solutions and that require the field to decay sufficiently fast at large distances. ■

We construct the canonical momenta Π^μ conjugate to C_μ from the definition

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 C_\mu)} = F^{\mu 0}, \tag{7.7}$$

which indeed vanishes when $\mu = 0$.

Proposition 7.3: *The Hamiltonian density for this theory is*

$$\mathcal{H} = -\frac{1}{2}\Pi_i \Pi^i + \frac{1}{2}(\partial_i C_j \partial^i C^j - \partial_i C_j \partial^j C^i) - \frac{1}{2} m^2 C_i C^i - \frac{1}{2} m^2 (C_0)^2 - C_0 \partial_i \Pi^i. \tag{7.8}$$

Proof.—The Hamiltonian density is obtained from the Legendre transform $\mathcal{H} = \Pi^\mu \dot{C}_\mu - \mathcal{L}$, which yields

$$\mathcal{H} = \Pi^i \dot{C}_i + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 C_\mu C^\mu. \tag{7.9}$$

This is still expressed in terms of time derivatives of the fields $\dot{C}_\mu$, which we need to eliminate in favour of the conjugate momenta. First note that $\Pi^i = F^{i0} = \partial^i C^0 - \partial^0 C^i$, which can be rearranged to read

$$\dot{C}_i = \partial_i C_0 - \Pi_i. \tag{7.10}$$

Next, we expand

$$\begin{aligned} F_{\mu\nu} F^{\mu\nu} &= 2 F_{0i} F^{0i} + F_{ij} F^{ij} \\ &= 2\Pi_i \Pi^i + 2(\partial_i C_j \partial^i C^j - \partial_i C_j \partial^j C^i). \end{aligned} \tag{7.11}$$

Putting everything together, we get

$$\mathcal{H} = -\frac{1}{2}\Pi_i \Pi^i + \frac{1}{2}(\partial_i C_j \partial^i C^j - \partial_i C_j \partial^j C^i) - \frac{1}{2} m^2 C_i C^i - \frac{1}{2} m^2 (C_0)^2 + \Pi^i \partial_i C_0. \tag{7.12}$$

The momentum term looks a little funny with the minus sign sitting in front of it, but this just has to do with our signature convention $(+,-,-,-)$ for the Minkowski metric. As a final step, we use the product rule to write the last term as $\Pi^i \partial_i C_0 = \partial_i(\Pi^i C_0) - C_0 \partial_i \Pi^i$. Under reasonable assumptions, the boundary term $\partial_i(\Pi^i C_0)$ vanishes when we integrate over all space to form the Hamiltonian $H = \int \mathrm{d}^3\mathbf{x}\, \mathcal{H}$, so can be discarded since it does not contribute to the equations of motion. Doing so gives us the desired result. ■

[11] Also sometimes called the inhomogeneous screened Poisson equation.

As a sanity check, observe that C_0 appears with no conjugate momentum in Eq. (7.8), thus it is serving as a Lagrange multiplier. It enforces the condition $C_0 = -(\partial_i \Pi^i)/m^2$. Plugging the definition $\Pi^i = F^{i0} = \partial^i C_0 - \dot{C}^i$ into this condition returns Eq. (7.4).

Question 8

Consider a scalar field theory in $d = n + 1$ dimensions with action

$$S = \int \mathrm{d}^d x \, \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - g\phi^p \right), \tag{8.1}$$

where m, g and p are constants. We are interested in determining the constraints on these constants that permit the theory to be scale invariant. If we take the viewpoint of an active transformation, this involves scaling

$$\phi(x) \to \phi'(x) = \lambda^{-D} \phi(\lambda^{-1} x), \tag{8.2}$$

where D is called the (classical) scaling dimension of the field.

Proposition 8.1: *The derivative terms are scale invariant only if*

$$D = \frac{d-2}{2} = \frac{n-1}{2}. \tag{8.3}$$

Proof.—Under this scaling transformation, we have that

$$\frac{\partial \phi(x)}{\partial x^\mu} \to \frac{\partial \phi'(x)}{\partial x^\mu} = \lambda^{-D} \frac{\partial \phi(\lambda^{-1} x)}{\partial x^\mu} = \lambda^{-D-1} \frac{\partial \phi(y)}{\partial y^\mu}. \tag{8.4}$$

In the last step, we used the chain rule and defined $y^\mu = \lambda^{-1} x^\mu$. This tells us that $\partial_\mu \phi(x) \to \lambda^{-D-1} \partial_\mu \phi(y)$, hence the derivative terms in the action transform as

$$\begin{aligned} \frac{1}{2} \int \mathrm{d}^d x \, \partial_\mu \phi(x) \partial^\mu \phi(x) &\to \lambda^{-2(D+1)} \frac{1}{2} \int \mathrm{d}^d x \, \partial_\mu \phi(y) \partial^\mu \phi(y) \\ &= \lambda^{d-2(D+1)} \frac{1}{2} \int \mathrm{d}^d y \, \partial_\mu \phi(y) \partial^\mu \phi(y). \end{aligned} \tag{8.5}$$

We arrive at the second line by transforming the measure into the new coordinates, $\mathrm{d}^d x = \lambda^d \, \mathrm{d}^d y$. Whether we call the coordinates x or y ultimately doesn't matter if they are being integrated over, hence the action is invariant under this rescaling provided

$$d - 2(D+1) = 0. \tag{8.6}$$

This can be solved for D to return the desired result. ■

Proposition 8.2: *Scale invariance requires the scalar to be massless ($m = 0$). If $g \neq 0$, it also requires*

$$p = \frac{2d}{d-2} = \frac{2n+2}{n-1}. \tag{8.7}$$

Proof.—All we have to do is repeat the same steps earlier. For the mass term, we find

$$\int \mathrm{d}^d x \, m^2 \phi^2(x) \to \lambda^{-2D} \int \mathrm{d}^d x \, m^2 \phi^2(y) = \lambda^{d-2D} \int \mathrm{d}^d y \, m^2 \phi^2(y). \tag{8.8}$$

This is scale invariant if $d = 2D$, but we already fixed $D = (d-2)/2$, hence scale invariance requires

$m = 0$. For the remaining power-law term,

$$\int \mathrm{d}^d x \; g\phi^p(x) \to \lambda^{d-Dp} \int \mathrm{d}^d y \; g\phi^p(y), \tag{8.9}$$

hence the theory is scale invariant if $d - Dp = 0$, which can be solved to give the desired result. ■

Proposition 8.3: *Scale invariance leads to the corresponding Noether current*

$$D^\mu = \left(\frac{d-2}{2}\right)\phi\partial^\mu\phi + \left(x^\nu\partial^\mu\phi - \frac{1}{2}x^\mu\partial^\nu\phi\right)\partial_\nu\phi + x^\mu g\phi^{2d/(d-2)}. \tag{8.10}$$

Proof.—We write $\lambda \simeq 1 - \epsilon$ such that an infinitesimal scaling transformation yields

$$\phi(x) \to \lambda^{-D}\phi(\lambda^{-1}x) = \phi(x) + \epsilon\left[D\phi(x) + x^\mu\partial_\mu\phi(x)\right], \tag{8.11}$$

from which we read off the infinitesimal change $\delta\phi = (D + x^\mu\partial_\mu)\phi$, having dropped the factor of ϵ as usual. The Lagrangian must similarly transform as

$$\mathcal{L}(x) \to \lambda^{-\ell}\mathcal{L}(\lambda^{-1}x) = \mathcal{L}(x) + \epsilon(\ell + x^\mu\partial_\mu)\mathcal{L}(x), \tag{8.12}$$

where ℓ is the scaling dimension of the Lagrangian. We determine the value of ℓ by noting that

$$\int \mathrm{d}^d x \; \mathcal{L}(x) \to \lambda^{d-\ell}\int \mathrm{d}^d y \; \mathcal{L}(y) \tag{8.13}$$

is scale invariant only if $\ell = d$. Thus, dropping the overall factor of ϵ,

$$\delta\mathcal{L} = (d + x^\mu\partial_\mu)\mathcal{L} = \partial_\mu(x^\mu\mathcal{L}), \tag{8.14}$$

since $\partial_\mu x^\mu = \delta^\mu_\mu = d$. We read off $F^\mu = x^\mu\mathcal{L}$. These tell us that the Noether current is

$$\begin{aligned} D^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - F^\mu \\ &= \partial^\mu\phi\,(D\phi + x^\nu\partial_\nu\phi) - x^\mu\left(\frac{1}{2}\partial_\nu\phi\partial^\nu\phi - g\phi^p\right), \end{aligned} \tag{8.15}$$

which yields the desired result upon further simplification. ■

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[3] D. N. Blaschke, F. Gieres, M. Reboud, and M. Schweda, "The energy–momentum tensor(s) in classical gauge theories", Nucl. Phys. **B912**, 192 (2016), arXiv:1605.01121.