Quantum Field Theory: Example Sheet 2

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Question 1

In this example sheet, Questions 1–6 concern the real scalar field ϕ governed by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \tag{1.1}$$

Proposition 1.1: *The normal-ordered four-momentum operator is*

$$:P^\mu: = \int \frac{\mathrm{d}^3p}{(2\pi)^3} p^\mu a_p^\dagger a_p. \tag{1.2}$$

Proof.—We know from Noether's theorem, and the fact that Eq. (1.1) is invariant under spacetime translations, that we have the conserved charges $P^\mu = \int \mathrm{d}^3x\, T^{0\mu}$, where $T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$. Noting that the conjugate momentum field

$$\pi(x) = \dot{\phi}(x) \tag{1.3}$$

by definition, we can read off the time and space components

$$P^0(t) = \int \mathrm{d}^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right), \tag{1.4}$$

$$P^i(t) = \int \mathrm{d}^3x\, \pi\partial^i\phi. \tag{1.5}$$

These quantities are still formally functions of time, but since they are conserved Noether charges, they are the same when evaluated at any given time. Naturally, the most convenient choice is $t = 0$, so we define the (classical) Hamiltonian $H \equiv P^0(0)$, and likewise the 3-momentum of the field is $P^i \equiv P^i(0)$.

We now move to quantize this theory. Our starting point is to use the Schrödinger-picture fields

$$\phi(\mathbf{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p e^{i\mathbf{p}\cdot\mathbf{x}} + a_p^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right), \tag{1.6a}$$

$$\pi(\mathbf{x}) = -i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{E_p}{2}}\left(a_p e^{i\mathbf{p}\cdot\mathbf{x}} - a_p^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right), \tag{1.6b}$$

where $E_p = \sqrt{\mathbf{p}^2 + m^2}$. The spatial derivative of $\phi(\mathbf{x})$ is

$$\partial_i\phi(\mathbf{x}) = -i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}p_i\left(a_p e^{i\mathbf{p}\cdot\mathbf{x}} - a_p^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right). \tag{1.7}$$

Note that there is a small subtlety with minus signs that we have to be careful of because of our choice of metric signature. The dot product $\mathbf{p}\cdot\mathbf{x} = \delta_{ij}p^i x^j$ is with respect to the Kronecker delta, so $\partial_i(\mathbf{p}\cdot\mathbf{x}) = \delta_{ij}p^j$. When we lower the index, we write $\delta_{ij}p^j = -\eta_{ij}p^j = -p_i$; it is very easy to forget this minus sign.[1]

Substitute these into the expression for P^i to get

$$P^i = \int \mathrm{d}^3x\, \pi(\mathbf{x})\partial^i\phi(\mathbf{x})$$

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[1] Things are much more straightforward when we work with 4-vectors; see the discussion around Eq. (4.5).

$$= -\int d^3x \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\sqrt{\frac{E_p}{2}}\left(a_p e^{i\mathbf{p}\cdot\mathbf{x}} - a_p^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \times \frac{1}{\sqrt{2E_q}} q_i \left(a_q e^{i\mathbf{q}\cdot\mathbf{x}} - a_q^\dagger e^{-i\mathbf{q}\cdot\mathbf{x}}\right)$$

$$= -\frac{1}{2}\int d^3x \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\sqrt{\frac{E_p}{E_q}} q^i \left[\left(a_p a_q + a_p^\dagger a_q^\dagger\right) e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} - \left(a_p a_q^\dagger + a_p^\dagger a_q\right) e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\right]. \tag{1.8}$$

In obtaining the last line, we use the freedom to relabel the integration variable $\mathbf{x} \to -\mathbf{x}$ in two of the four terms in square brackets to simplify the expression. Integrating over $\mathbf{x}$ now pulls down delta functions, i.e., $\int d^3x\, e^{i(\mathbf{p}\pm\mathbf{q})\cdot\mathbf{x}} = (2\pi)^3\delta^{(3)}(\mathbf{p}\pm\mathbf{q})$. Integrate over $\mathbf{q}$ to obtain

$$P^i = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} p^i \left(a_p a_{-p} + a_p^\dagger a_{-p}^\dagger + a_p a_p^\dagger + a_p^\dagger a_p\right). \tag{1.9}$$

The first two terms $p^i a_p a_{-p}$ and $p^i a_p^\dagger a_{-p}^\dagger$ are odd in $\mathbf{p}$ since the operators commute, thus they vanish when we integrate over all values of the momenta. Only the last two terms remain, which upon normal ordering yields the desired result,

$$:P^i: = \int \frac{d^3p}{(2\pi)^3}\, p^i a_p^\dagger a_p. \tag{1.10}$$

We repeat similar steps to obtain the Hamiltonian. Substitute Eqs. (1.6) and (1.7) into the expression for the Hamiltonian to obtain[2]

$$H = \int d^3x \left(\frac{1}{2}\pi^2(\mathbf{x}) + \frac{1}{2}\delta^{ij}\partial_i\phi(\mathbf{x})\partial_j\phi(\mathbf{x}) + \frac{1}{2}m^2\phi^2(\mathbf{x})\right)$$

$$= \frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\frac{1}{E_p}\left[(-E_p^2 + \mathbf{p}^2 + m^2)(a_p a_{-p} + a_p^\dagger a_{-p}^\dagger) + (E_p^2 + \mathbf{p}^2 + m^2)(a_p a_p^\dagger + a_p^\dagger a_p)\right]. \tag{1.11}$$

The term proportional to $(-E_p^2 + \mathbf{p}^2 + m^2)$ vanishes because $E_p^2 = \mathbf{p}^2 + m^2$, leaving us with only the other term. After normal ordering, we get

$$:H: = \int \frac{d^3p}{(2\pi)^3}\, E_p a_p^\dagger a_p. \tag{1.12}$$

Recalling that $H \equiv P^0$, Eqs. (1.10) and (1.12) together yield the desired result for $:P^\mu:$. ■

Lemma 1.1: *The Heisenberg-picture fields $\phi(x) \equiv \phi(t, \mathbf{x})$ and $\pi(x) = \pi(t, \mathbf{x})$ are given by*

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right), \tag{1.13a}$$

$$\pi(x) = -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_p}{2}}\left(a_p e^{-ip\cdot x} - a_p^\dagger e^{ip\cdot x}\right). \tag{1.13b}$$

Proof.—Any (time-independent) Schrödinger-picture operator $\mathcal{O}_S$ can be transformed into a Heisenberg-picture operator via the definition

$$\mathcal{O}_H(t) = e^{iHt}\mathcal{O}_S e^{-iHt}. \tag{1.14}$$

Differentiate this to get

$$\frac{d}{dt}\mathcal{O}_H = e^{-iHt}[iH, \mathcal{O}_S]e^{-iHt} = i[H, \mathcal{O}_H], \tag{1.15}$$

having used the fact that $\mathcal{O}_S$ is time-independent, and $\partial_t(e^{iHt}) = iHe^{iHt} = e^{iHt}iH$, since H necessarily commutes with itself. For the specific case of constructing Heisenberg-picture fields out of $\phi(\mathbf{x})$ or $\pi(\mathbf{x})$, what matters are the operators $e^{iHt}a_p e^{-iHt}$ and $e^{iHt}a_p^\dagger e^{-iHt}$, which we would like to simplify.

[2] See the discussion around Eq. (2.22) of David Tong's notes [1] for intermediate steps.

Using Eq. (1.15) with $\mathcal{O}_S \to a_p$ and $\mathcal{O}_H \to a_p(t) \equiv e^{iHt} a_p e^{-iHt}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} a_p(t) = e^{iHt}[iH, a_p]e^{-iHt}. \tag{1.16}$$

We then use the commutation relations

$$[a_p, a_q^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [a_p, a_q] = [a_p^\dagger, a_q^\dagger] = 0 \tag{1.17}$$

combined with the definition of H in Eq. (1.12) to learn that

$$[H, a_p] = -E_p a_p, \quad [H, a_p^\dagger] = E_p a_p^\dagger. \tag{1.18}$$

These imply

$$\frac{\mathrm{d}}{\mathrm{d}t} a_p(t) = e^{iHt}(-iE_p a_p)e^{-iHt} = -iE_p a_p(t), \tag{1.19}$$

which is a standard differential equation with solution $a_p(t) = a_p(0)e^{-iE_p t}$. Since $a_p(0) = a_p$ by construction, we learn that

$$e^{iHt} a_p e^{-iHt} = a_p e^{-iE_p t}, \quad \text{and likewise} \quad e^{iHt} a_p^\dagger e^{-iHt} = a_p^\dagger e^{iE_p t}. \tag{1.20}$$

Finally, noting that $E_p = p_0$ allows us to write the dot product appearing in the exponentials as $-iE_p t + i\mathbf{p} \cdot \mathbf{x} = -ip_\mu x^\mu = -ip \cdot x$. This completes the proof. ■

Proposition 1.2: *The Heisenberg-picture field $\phi(x)$ satisfies the commutation relation*

$$[P^\mu, \phi(x)] = -i\partial^\mu \phi(x). \tag{1.21}$$

Proof.—Evaluate the LHS to obtain

$$\begin{aligned} [P^\mu, \phi(x)] &= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{\mathrm{d}^3q}{(2\pi)^3} \left[p^\mu a_p^\dagger a_p, \frac{1}{\sqrt{2E_q}} \left(a_q e^{-iq\cdot x} + a_q^\dagger e^{iq\cdot x} \right) \right] \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{\mathrm{d}^3q}{(2\pi)^3} \frac{p^\mu}{\sqrt{2E_q}} \left([a_p^\dagger a_p, a_q] e^{-iq\cdot x} + [a_p^\dagger a_p, a_q^\dagger] e^{iq\cdot x} \right). \end{aligned} \tag{1.22}$$

Now use the commutation relations in Eq. (1.17) to simplify this to

$$\begin{aligned} [P^\mu, \phi(x)] &= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{\mathrm{d}^3q}{(2\pi)^3} \frac{p^\mu}{\sqrt{2E_q}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left(-a_p e^{-iq\cdot x} + a_p^\dagger e^{iq\cdot x} \right) \\ &= -\int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{p^\mu}{\sqrt{2E_p}} \left(a_p e^{-ip\cdot x} - a_p^\dagger e^{ip\cdot x} \right). \end{aligned} \tag{1.23}$$

One can then verify that this last line is indeed equal to the derivative $-i\partial^\mu \phi(x)$. ■

Question 2

Proposition 2.1: *The Heisenberg-picture fields satisfy $\dot{\phi}(x) = i[H, \phi(x)] = \pi(x)$.*

Proof.—Recall that the Hamiltonian for a free scalar field is

$$H = \int \mathrm{d}^3x \left(\frac{1}{2}\pi^2(x) + \frac{1}{2}[\nabla\phi(x)]^2 + \frac{1}{2}m^2\phi^2(x) \right), \tag{2.1}$$

and that it is a conserved charge so can be evaluated at any time we like. We choose to evaluate it at the time $t = x^0$ to make use of the equal-time commutation relations

$$[\phi(t,\mathbf{x}), \pi(t,\mathbf{x}')] = i\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \quad [\phi(t,\mathbf{x}), \phi(t,\mathbf{x}')] = [\pi(t,\mathbf{x}), \pi(t,\mathbf{x}')] = 0. \tag{2.2}$$

Since ϕ commutes with itself, the only contribution comes from

$$i[H, \phi(x)] = i \int \mathrm{d}^3x' \left(\frac{1}{2} \left[\pi^2(t,\mathbf{x}'), \phi(t,\mathbf{x})\right] \right) = \pi(x). \tag{2.3}$$

This completes the proof, since the other relations—namely, $\dot{\phi}(x) = \pi(x)$ and $\dot{\phi}(x) = i[H, \phi(x)]$—are true by definition; cf. Eqs. (1.3) and (1.15), and also Eq. (1.21). ■

Proposition 2.2: *The fields also satisfy* $\dot{\pi}(x) = i[H, \pi(x)] = (\nabla^2 - m^2)\phi(x)$.

Proof.—The first equality $\dot{\pi}(x) = i[H, \pi(x)]$ is true by definition, cf. Eq. (1.15). Let us verify the second equality. We find

$$i[H, \pi(x)] = \frac{i}{2} \int \mathrm{d}^3x' \left(\left[(\nabla\phi(t,\mathbf{x}'))^2, \pi(t,\mathbf{x})\right] + m^2 \left[\phi^2(t,\mathbf{x}'), \pi(t,\mathbf{x})\right] \right). \tag{2.4}$$

We can simplify the first commutator by noting

$$\begin{aligned} \frac{1}{2}\left[(\nabla\phi(t,\mathbf{x}'))^2, \pi(t,\mathbf{x})\right] &= \nabla\phi(t,\mathbf{x}') \cdot [\nabla\phi(t,\mathbf{x}'), \pi(t,\mathbf{x})] \\ &= \delta^{ij} \frac{\partial\phi(t,\mathbf{x}')}{\partial x'^i} \frac{\partial}{\partial x'^j} [\phi(t,\mathbf{x}'), \pi(t,\mathbf{x})] \\ &= i\delta^{ij} \frac{\partial\phi(t,\mathbf{x}')}{\partial x'^i} \frac{\partial}{\partial x'^j} \delta^{(3)}(\mathbf{x}'-\mathbf{x}). \end{aligned} \tag{2.5}$$

Plug this back into Eq. (2.4) and integrate by parts to get

$$i[H, \pi(x)] = \int \mathrm{d}^3x' \left(\nabla^2\phi(t,\mathbf{x}')\delta^{(3)}(\mathbf{x}'-\mathbf{x}') + \frac{i}{2} m^2 [\phi^2(t,\mathbf{x}'), \pi(t,\mathbf{x})] \right). \tag{2.6}$$

Simplifying the remaining commutator is straightforward, and yields the desired result. ■

Since $\dot{\phi} = \pi$ and $\dot{\pi} = (\nabla^2 - m^2)\phi$, it is clear that ϕ satisfies the Klein–Gordon equation, as it should.

Question 3

Proposition 3.1: *The momentum eigenstate* $|p\rangle = \sqrt{2E_p} a_p^\dagger |0\rangle$ *satisfies* $\langle 0 | \phi(x) | p \rangle = e^{-ip\cdot x}$.

Proof.—It will be instructive to instead prove the complex conjugate statement, $\langle p | \phi(x) | 0 \rangle = e^{ip\cdot x}$. We start by determining

$$\begin{aligned} \phi(x)|0\rangle &= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right)|0\rangle \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p^\dagger e^{ip\cdot x}|0\rangle \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p} e^{ip\cdot x}|p\rangle. \end{aligned} \tag{3.1}$$

In the second line, we have used the fact that the vacuum is defined by the condition $a_p|0\rangle = 0\ \forall p$. From the commutation relations in Eq. (1.17), the momentum eigenstates are found to satisfy the normalization condition

$$\langle p | q \rangle = (2\pi)^3 2E_p \delta^{(3)}(\mathbf{p}-\mathbf{q}), \tag{3.2}$$

hence we obtain $\langle p\,|\phi(x)|\,0\rangle = e^{ip\cdot x}$ as desired. ■

Aside 3.1: This result looks very similar to one in standard quantum mechanics, $\langle \mathbf{p}\,|\mathbf{x}\rangle \propto e^{-i\mathbf{p}\cdot\mathbf{x}}$. Accordingly, we can interpret $\phi(x)\,|0\rangle \equiv |x\rangle$ as a state with a single particle localized at the spacetime point x. As we should be accustomed to by now, the last line of Eq. (3.1) is telling us that a particle localized at a single point is equivalent to a superposition of momentum eigenstates.

Question 4

Recall from Example Sheet 1 that a Lorentz-invariant theory conserves the classical angular momentum of the field,

$$Q_i = \frac{1}{2}\epsilon_{ijk}\int \mathrm{d}^3x\,\left(x^j T^{0k} - x^k T^{0j}\right). \tag{4.1}$$

For a free scalar field with energy–momentum tensor $T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$, this has the explicit form

$$Q_i = \frac{1}{2}\epsilon_{ijk}\int \mathrm{d}^3x\;\dot{\phi}\left(x^j\partial^k\phi - x^k\partial^j\phi\right) = \epsilon_{ijk}\int \mathrm{d}^3x\;\pi(x)\,x^j\partial^k\phi(x). \tag{4.2}$$

Along with the definition $\dot{\phi} = \pi$, we use the antisymmetry property of ϵ_{ijk} and the freedom to relabel dummy indices to obtain the last expression.

Proposition 4.1: *The normal-ordered quantum operator Q_i is*[3]

$$Q_i = \frac{i}{2}\epsilon_{ijk}\int \frac{\mathrm{d}^3p}{(2\pi)^3}a_p^\dagger\left(p^j\frac{\partial}{\partial p_k} - p^k\frac{\partial}{\partial p_j}\right)a_p. \tag{4.3}$$

Proof.—As we saw in an earlier question, we have to worry about minus signs when there are dot products over three-dimensional objects like $\mathbf{p}\cdot\mathbf{x}$. For this reason, we keep Q_i as a function of some arbitrary time t until it stops being convenient. We substitute in the Heisenberg fields of Eqs. (1.13) to get

$$\begin{aligned}\epsilon_{ijk}\int \mathrm{d}^3x\;\pi x^j\partial^k\phi = -\frac{i}{2}\epsilon_{ijk}\int \mathrm{d}^3x\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{E_p}{E_q}}\left(a_p e^{-ip\cdot x} - a_p^\dagger e^{ip\cdot x}\right)\\ \times x^j(-iq^k)\left(a_q e^{-iq\cdot x} - a_q^\dagger e^{iq\cdot x}\right).\end{aligned} \tag{4.4}$$

We would like to simplify this by integrating over $\mathbf{x}$, but the factor of x^j makes this problematic. We deal with this by realizing that

$$\frac{\partial}{\partial q_\mu}e^{-iq\cdot x} = -ix^\mu e^{-iq\cdot x}, \tag{4.5}$$

such that

$$Q_i = -\frac{i}{2}\epsilon_{ijk}\int \mathrm{d}^3x\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{E_p}{E_q}}\left(a_p e^{-ip\cdot x} - a_p^\dagger e^{ip\cdot x}\right)q^k\left(a_q\frac{\partial}{\partial q^j}e^{-iq\cdot x} + a_q^\dagger\frac{\partial}{\partial q^j}e^{iq\cdot x}\right). \tag{4.6}$$

Now use the freedom to choose $t = 0$ (since Q_i is a conserved charge) and integrate over $\mathbf{x}$ to find

$$Q_i = \frac{i}{2}\epsilon_{ijk}\int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{E_p}{E_q}}q^k\left[(a_p^\dagger a_q^\dagger - a_p a_q)\frac{\partial}{\partial q_j}\delta^{(3)}(\mathbf{p}+\mathbf{q}) + (a_p^\dagger a_q - a_p a_q^\dagger)\frac{\partial}{\partial q_j}\delta^{(3)}(\mathbf{p}-\mathbf{q})\right]. \tag{4.7}$$

[3] This differs from the result given in the question sheet by a minus sign. I think I am correct, but if I have committed a sign error somewhere and you spot it, please let me know.

The natural thing to do next is to move the differential operator $\partial/\partial q_j$ off the delta function by integrating by parts, so that we can integrate over $\mathbf{p}$. Doing it this way turns out to be rather tedious. Instead, let us define another differential operator

$$L_i^{(\mathbf{q})} \equiv i\epsilon_{ijk} q^k \frac{\partial}{\partial q_j}, \tag{4.8}$$

which one should recognize as the angular momentum $\mathbf{L}^i = (\mathbf{x} \times \mathbf{p})^i$ expressed in momentum space. The superscript on $L_i^{(\mathbf{q})}$ tells us that this is an operator acting on functions of $\mathbf{q}$. It is easy to convince ourselves that this operator satisfies the Leibniz rule

$$L_i^{(\mathbf{q})}[f(\mathbf{q})g(\mathbf{q})] = [L_i^{(\mathbf{q})} f(\mathbf{q})]g(\mathbf{q}) + f(\mathbf{q})[L_i^{(\mathbf{q})} g(\mathbf{q})] \tag{4.9}$$

for any two functions f, g. This means integration by parts also works for this operator, thus

$$\begin{aligned} Q_i &= \frac{1}{2}\int \frac{\mathrm{d}^3 p\, \mathrm{d}^3 q}{(2\pi)^3}\sqrt{\frac{E_p}{E_q}}\left[(a_p^\dagger a_q^\dagger - a_p a_q)L_i^{(\mathbf{q})}\delta^{(3)}(\mathbf{p}+\mathbf{q}) + (a_p^\dagger a_q - a_p a_q^\dagger)L_i^{(\mathbf{q})}\delta^{(3)}(\mathbf{p}-\mathbf{q})\right] \\ &= \frac{1}{2}\int \frac{\mathrm{d}^3 p\, \mathrm{d}^3 q}{(2\pi)^3}\sqrt{\frac{E_p}{E_q}}\left[\delta^{(3)}(\mathbf{p}+\mathbf{q})L_i^{(\mathbf{q})}(a_p a_q - a_p^\dagger a_q^\dagger) + \delta^{(3)}(\mathbf{p}-\mathbf{q})L_i^{(\mathbf{q})}(a_p a_q^\dagger - a_p^\dagger a_q)\right] \\ &= \frac{1}{2}\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\left(a_{-q}L_i^{(\mathbf{q})}a_q - a_{-q}^\dagger L_i^{(\mathbf{q})}a_q^\dagger + a_q L_i^{(\mathbf{q})}a_q^\dagger - a_q^\dagger L_i^{(\mathbf{q})}a_q\right), \end{aligned} \tag{4.10}$$

having integrated over $\mathbf{p}$ to obtain the last line. Note also that, in obtaining the second line, we have used the fact that

$$L_i^{(\mathbf{q})} f(\mathbf{q}^2) = 0 \tag{4.11}$$

for any spherically-symmetric function $f(\mathbf{q}^2)$.[4] In particular, this means $L_i^{(\mathbf{q})}E_q = 0$. Let's rename $\mathbf{q} \to \mathbf{p}$ and drop the superscripts on L_i (since it's clear it's acting on $\mathbf{p}$) to get

$$Q_i = \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left(a_{-p}L_i a_p - a_{-p}^\dagger L_i a_p^\dagger + a_p L_i a_p^\dagger - a_p^\dagger L_i a_p\right). \tag{4.12}$$

It turns out that the first two terms above are zero. To see this, note that L_i is unchanged under the transformation $\mathbf{p} \to -\mathbf{p}$, which implies

$$a_{-p}L_i a_p \xrightarrow{①} -a_p L_i a_{-p} \xrightarrow{②} -a_{-p}L_i a_p, \tag{4.13}$$

having integrated by parts in step ①, and then using the freedom to transform $\mathbf{p} \to -\mathbf{p}$ under the integral in step ②. Since this is odd in $\mathbf{p}$, it vanishes when we integrate over all values of the momenta. Likewise, the same arguments apply to the $a_{-p}^\dagger L_i a_p^\dagger$ term. After normal ordering, we are left with

$$:\!Q_i\!: = \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[(L_i a_p^\dagger)a_p - a_p^\dagger L_i a_p\right] = -\int \frac{\mathrm{d}^3 p}{(2\pi)^3} a_p^\dagger L_i a_p, \tag{4.14}$$

having integrated by parts to obtain the second expression. This is the desired result. ■

Proposition 4.2: *For a momentum eigenstate* $|p\rangle$*, its angular momentum is* $Q_i|p\rangle = L_i|p\rangle$*.*

Proof.—Let us act on the state $|p\rangle = \sqrt{2E_p}a_p^\dagger|0\rangle$ with the operator Q_i. We find

$$Q_i|p\rangle = -\int \frac{\mathrm{d}^3 q}{(2\pi)^3} a_q^\dagger L_i^{(\mathbf{q})} a_q \sqrt{2E_p}a_p^\dagger|0\rangle = \int \frac{\mathrm{d}^3 q}{(2\pi)^3}[L_i^{(\mathbf{q})}\sqrt{2E_p}a_q^\dagger]a_q a_p^\dagger|0\rangle \tag{4.15}$$

[4]One sees this by noting that $\partial/\partial q_j f(\mathbf{q}^2)$ produces something proportional to q^j. Then $L_i^{(\mathbf{q})}f(\mathbf{q}^2) \propto \epsilon_{ijk}q^j q^k = 0$.

after integrating by parts. We can replace $a_q a_p^\dagger \to [a_q, a_p^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})$ in the integrand and integrate over $\mathbf{q}$ to find the desired result. ■

Proposition 4.2 tells us that the total angular momentum of the state $|p\rangle$ is due only to its motion in spacetime, i.e., there is no contribution from any intrinsic degrees of freedom. If the particle is massive, then we can boost the result to its rest frame, where we would have $Q_i\,|\mathbf{0}\rangle = 0$.

Aside 4.1: One might wonder what it means to take the derivative $\partial/\partial\mathbf{p}$ of a momentum eigenstate $|p\rangle$. Momentum eigenstates are nice and simple objects to work with, but we can be a bit more rigorous here by working with a wavepacket

$$|\Phi\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\Phi(p)\,|p\rangle\,, \tag{4.16}$$

where $\Phi(p)$ is some function that might be peaked around a particular momentum value. These are really what we create and observe in experiments. One can then show that the expectation value of the angular momentum for this wavepacket is

$$\langle\Phi\,|Q_i|\,\Phi\rangle = -\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\Phi^*(p)L_i\Phi(p). \tag{4.17}$$

Notice that the differential operator L_i is now just acting on a good ol' function $\Phi(p)$.

Question 5

Proposition 5.1: *The normal-ordered product* $:\!\phi(x_1)\phi(x_2)\!:$ *is symmetric under the interchange* $x_1 \leftrightarrow x_2$.

Proof.—We decompose the scalar field as $\phi(x) = \phi^+(x) + \phi^-(x)$, where

$$\phi^+(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}a_p e^{-ip\cdot x}, \quad \phi^-(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}a_p^\dagger e^{ip\cdot x}. \tag{5.1}$$

For brevity, let us write $\phi_1 \equiv \phi(x_1)$ and $\phi_2 \equiv \phi(x_2)$ Then the normal-ordered product is

$$:\!\phi_1\phi_2\!: = :\!(\phi_1^+ + \phi_1^-)(\phi_2^+ + \phi_2^-)\!: = \phi_1^+\phi_2^+ + \phi_2^-\phi_1^+ + \phi_1^-\phi_2^+ + \phi_1^-\phi_2^-\,. \tag{5.2}$$

The commutation relations in Eq. (1.17) tell us that $[\phi_1^+, \phi_2^+] = [\phi_1^-, \phi_2^-] = 0$ (even when their times are not equal), hence the equation above is manfiestly symmetric under the interchange $x_1 \leftrightarrow x_2$. ■

Proposition 5.2: *The time-ordered product* $T[\phi(x_1)\phi(x_2)]$ *is symmetric under the interchange* $x_1 \leftrightarrow x_2$.

Proof.—By definition, the time-ordered product is

$$T[\phi(x_1)\phi(x_2)] = \theta(x_1^0 - x_2^0)\phi(x_1)\phi(x_2) + \theta(x_2^0 - x_1^0)\phi(x_2)\phi(x_1). \tag{5.3}$$

This is manifestly symmetric. ■

The Feynman propagator $\Delta_F(x_1 - x_2)$ is just the linear combination

$$\Delta_F(x_1 - x_2) \equiv T[\phi(x_1)\phi(x_2)] - :\!\phi(x_1)\phi(x_2)\!:, \tag{5.4}$$

hence it inherits the same symmetry property of its constituents.

Question 6

It will be convenient to begin by establishing more compact notation for this question. We write $\phi_i \equiv \phi(x_i)$ to denote the argument of the scalar field, $\Delta_{ij} \equiv \Delta_F(x_i - x_j)$ for the Feynman propagator, and $\theta_{ij} = \theta(t_i - t_j)$ for the Heaviside step function, where $t \equiv x^0$. Furthermore, let us also define $\theta_{ijk} = \theta_{ij}\theta_{jk}$. Note that these indices are never implicitly summed over. We are asked to verify that Wick's theorem holds for the 3-point correlation function, which written in this compact notation reads

$$T[\phi_1\phi_2\phi_3] = {:}\phi_1\phi_2\phi_3{:} + \phi_1\Delta_{23} + \phi_2\Delta_{31} + \phi_3\Delta_{12}. \tag{6.1}$$

Lemma 6.1: *The time-ordered product can be written as*

$$T[\phi_1\phi_2\phi_3] = \sum_{\sigma\in S_3} \theta_{\sigma(1)\sigma(2)\sigma(3)}\phi_{\sigma(1)}\phi_{\sigma(2)}\phi_{\sigma(3)}. \tag{6.2}$$

Proof.—We expand out the definition of the time-ordered product to obtain

$$T[\phi_1\phi_2\phi_3] = \theta_{123}\phi_1\phi_2\phi_3 + \theta_{213}\phi_2\phi_1\phi_3 + (4 \text{ other permutations}). \tag{6.3}$$

The group S_3 is exactly the set of all permutations of three elements, hence this expansion can be written succinctly according to Eq. (6.2). ■

Lemma 6.2: *The normal-ordered product is symmetric under the interchange of any two arguments,*

$${:}\phi_1\phi_2\phi_3{:} = {:}\phi_2\phi_1\phi_3{:} = {:}\phi_1\phi_3\phi_2{:} = {:}\phi_3\phi_2\phi_1{:}\,. \tag{6.4}$$

Proof.—We expand out the definition of the normal-ordered product to obtain

$$\begin{aligned} {:}\phi_1\phi_2\phi_3{:} = {} & \phi_1^+\phi_2^+\phi_3^+ \\ & + \phi_1^-\phi_2^+\phi_3^+ + \phi_2^-\phi_3^+\phi_1^+ + \phi_3^-\phi_1^+\phi_2^+ \\ & + \phi_1^-\phi_2^-\phi_3^+ + \phi_2^-\phi_3^-\phi_1^+ + \phi_3^-\phi_1^-\phi_2^+ \\ & + \phi_1^-\phi_2^-\phi_3^-\,. \end{aligned} \tag{6.5}$$

Since $[\phi_i^\pm, \phi_j^\pm] = 0$, this is manifestly symmetric under the interchange of any two arguments. ■

Lemma 6.3: *The normal-ordered product can be written as*

$${:}\phi_1\phi_2\phi_3{:} = \sum_{\sigma\in S_3} \theta_{\sigma(1)\sigma(2)\sigma(3)}\ {:}\phi_{\sigma(1)}\phi_{\sigma(2)}\phi_{\sigma(3)}{:}\,. \tag{6.6}$$

Proof.—As a generalization of the identity $\theta(x) + \theta(-x) = 1$, notice that if we "time-order the identity," we get

$$1 = T[1] = \sum_{\sigma\in S_3} \theta_{\sigma(1)\sigma(2)\sigma(3)}. \tag{6.7}$$

We can multiply the normal-ordered product by this identity to get

$$\begin{aligned} {:}\phi_1\phi_2\phi_3{:} &= T[1]\ {:}\phi_1\phi_2\phi_3{:} \\ &= \big(\theta_{123} + \theta_{213} + \dots\big)\ {:}\phi_1\phi_2\phi_3{:} \\ &= \theta_{123}\ {:}\phi_1\phi_2\phi_3{:} + \theta_{213}\ {:}\phi_2\phi_1\phi_3{:} + \dots, \end{aligned} \tag{6.8}$$

having used Lemma 6.2 in the last line. This completes the proof. ■

Theorem 6.1: (Special case of Wick's theorem) *We are now in a position to prove Eq.* (6.1).

Proof.—Subtract the time and normal-ordered products while using Lemmas 6.1 and 6.3 to find

$$T[\phi_1\phi_2\phi_3] - :\!\phi_1\phi_2\phi_3\!: = \sum_{\sigma\in S_3} \theta_{\sigma(1)\sigma(2)\sigma(3)}\left\{\phi_{\sigma(1)}\phi_{\sigma(2)}\phi_{\sigma(3)} - :\!\phi_{\sigma(1)}\phi_{\sigma(2)}\phi_{\sigma(3)}\!: \right\}. \tag{6.9}$$

Now expand $\phi_i = \phi_i^+ + \phi_i^-$ within the curly brackets. Note that terms with three factors of ϕ^+ or three factors of ϕ^- will cancel each other, leaving us with

$$\begin{aligned}\phi_{\sigma(1)}\phi_{\sigma(2)}\phi_{\sigma(3)} - :\!\phi_{\sigma(1)}\phi_{\sigma(2)}\phi_{\sigma(3)}\!: &= \left[\phi^+_{\sigma(1)}, \phi^-_{\sigma(2)}\right]\phi^+_{\sigma(3)} + \left[\phi^+_{\sigma(1)}\phi^+_{\sigma(2)}, \phi^-_{\sigma(3)}\right] \\ &\quad + \phi^-_{\sigma(1)}\left[\phi^+_{\sigma(2)}, \phi^-_{\sigma(3)}\right] + \left[\phi^+_{\sigma(1)}, \phi^-_{\sigma(2)}\phi^-_{\sigma(3)}\right]. \end{aligned} \tag{6.10}$$

We now expand the brackets in the second and fourth terms using $[AB, C] = A[B, C] + [A, C]B$, and further use the fact $[\phi_i^+, \phi_j^-]$ is just a number to find

$$\begin{aligned} &T[\phi_1\phi_2\phi_3] - :\!\phi_1\phi_2\phi_3\!: \\ &= \sum_{\sigma\in S_3} \theta_{\sigma(1)\sigma(2)\sigma(3)}\left\{\phi_{\sigma(1)}\left[\phi^+_{\sigma(2)}, \phi^-_{\sigma(3)}\right] + \phi_{\sigma(2)}\left[\phi^+_{\sigma(1)}, \phi^-_{\sigma(3)}\right] + \phi_{\sigma(3)}\left[\phi^+_{\sigma(1)}, \phi^-_{\sigma(2)}\right]\right\} \\ &= \sum_{\sigma\in S_3}\left(\theta_{\sigma(1)\sigma(2)\sigma(3)} + \theta_{\sigma(2)\sigma(1)\sigma(3)} + \theta_{\sigma(2)\sigma(3)\sigma(1)}\right)\phi_{\sigma(1)}\left[\phi^+_{\sigma(2)}, \phi^-_{\sigma(3)}\right], \end{aligned} \tag{6.11}$$

having relabelled the dummy indices to arrive at the second line. The last thing we have to do is simplify the sum of θ's. To do this, recall that $\theta_{ijk} = \theta_{ij}\theta_{jk}$. Suppressing the σ's for simplicity, we can write

$$\begin{aligned}\theta_{123} + \theta_{213} + \theta_{231} &= \theta_{12}\theta_{23} + \theta_{21}\theta_{13} + \theta_{23}\theta_{31} \\ &= \theta_{12}\theta_{23} + \theta_{21}\theta_{13}\theta_{23} + \theta_{23}\theta_{31}. \end{aligned} \tag{6.12}$$

To get the second line, notice that $\theta_{21}\theta_{13}$ is nonvanishing only when $t_2 > t_1 > t_3$. Since this is already imposing the condition $t_2 > t_3$, nothing changes if we multiply this by an extra factor of θ_{23}. This gives us

$$\begin{aligned}\theta_{123} + \theta_{213} + \theta_{231} &= \theta_{23}(\theta_{12} + \theta_{21}\theta_{13} + \theta_{31}) \\ &= \theta_{23}(\theta_{12} - \theta_{12}\theta_{13} + \theta_{13} + \theta_{31}) \\ &= \theta_{12}\theta_{23} - \theta_{12}\theta_{23}\theta_{13} + \theta_{23}, \end{aligned} \tag{6.13}$$

having used the identity $\theta_{ij} + \theta_{ji} = 1$ in both the second and third lines. Finally, we repeat the same argument as before to show that $\theta_{12}\theta_{23}\theta_{13} = \theta_{12}\theta_{23}$, which implies

$$\theta_{123} + \theta_{213} + \theta_{231} = \theta_{23}. \tag{6.14}$$

Plug this back into Eq. (6.11) to get

$$T[\phi_1\phi_2\phi_3] - :\!\phi_1\phi_2\phi_3\!:= \sum_{\sigma\in S_3}\phi_{\sigma(1)}\left\{\theta_{\sigma(2)\sigma(3)}\left[\phi^+_{\sigma(2)}, \phi^-_{\sigma(3)}\right]\right\}. \tag{6.15}$$

Let us focus on the terms with $\sigma(1) = 1$. The terms in the sum proportional to ϕ_1 are

$$\begin{aligned}\phi_1\sum_{\sigma\in S_2}\theta_{\sigma(2)\sigma(3)}\left[\phi^+_{\sigma(2)}, \phi^-_{\sigma(3)}\right] &= \phi_1\sum_{\sigma\in S_2}\theta_{\sigma(2)\sigma(3)}\left(\phi_{\sigma(2)}\phi_{\sigma(3)} - :\!\phi_{\sigma(2)}\phi_{\sigma(3)}\!:\right) \\ &= \phi_1\left(T[\phi_2\phi_3] - :\!\phi_2\phi_3\!:\right). \end{aligned} \tag{6.16}$$

This follows by essentially running previous parts of our proof in reverse, but applied to the case of two fields. In the second line, the object in brackets is just the Feynman propagator Δ_{23} by Wick's theorem, so we get $\phi_1\Delta_{23}$. Repeating the same steps for $\sigma(1) = 2$ or 3, we get the other two terms, $\phi_2\Delta_{31}$ and $\phi_3\Delta_{12}$, and hence we recover the desired result in Eq. (6.1). ■

Question 7

We wish to study the vacuum-to-vacuum amplitude $\langle 0\,|S|\,0\rangle$ for ϕ^4 theory. Expanded as an asymptotic series in λ,

$$\begin{aligned}\langle 0\,|S|\,0\rangle &= \langle 0|\,T\exp\left(-\frac{i\lambda}{4!}\int \mathrm{d}^4x\;\phi^4(x)\right)|0\rangle \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i\lambda}{4!}\right)^n\int \mathrm{d}^4x_1\cdots\int \mathrm{d}^4x_n\;\langle 0|\,T\phi^4(x_1)\cdots\phi^4(x_n)\,|0\rangle\,. \end{aligned}\tag{7.1}$$

The $n = 0$ term is trivial, and just returns $\langle 0\,|0\,\rangle = 1$.

O(λ) term — Consider the $n = 1$ term, which reads

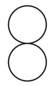
$$\langle 0\,|S|\,0\rangle \supset -i\frac{\lambda}{4!}\int \mathrm{d}^4x\,\langle 0|\,T\phi\phi\phi\phi\,|0\rangle = -i\frac{\lambda}{4!}(\mathbf{3})\int \mathrm{d}^4x\;\Delta_F^2(0),\tag{7.2}$$

having used Wick's theorem in the last step. The combinatorial factor of $\mathbf{3}$—highlighted in bold—comes from the total number of possible Wick contractions that lead to the same result,

$$\langle 0\,|T\phi\phi\phi\phi|\,0\rangle = \phi\phi\phi\phi + \phi\phi\phi\phi + \phi\phi\phi\phi = \mathbf{3}\,\Delta_F^2(0).\tag{7.3}$$

This can be represented diagrammatically by writing

$$-i\frac{\lambda}{4!}(\mathbf{3})\int \mathrm{d}^4x\;\Delta_F^2(0) =$$

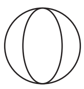


. (7.4)

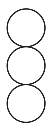
Notice that we should always include the overall numerical factor in the definition of the Feynman diagram.

O(λ^2) terms — We now turn to the $n = 2$ terms. In practice, we draw all the possible Feynman diagrams first to enumerate the different possibilities of Wick contractions, and only then do we figure out the combinatorial factor out front. There are three topologically distinct diagrams at the $n = 2$ level,

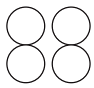
$\langle 0\,|S|\,0\rangle \supset$



$+$

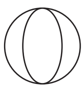


$+$



. (7.5)

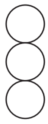
- The first of these diagrams yields



$$= \frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^2(\mathbf{24})\int \mathrm{d}^4x\,\mathrm{d}^4y\;\Delta_F^4(x-y).\tag{7.6}$$

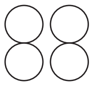
  To count the combinatorial factor of $\mathbf{24}$, start by choosing one leg on the bottom vertex (it doesn't matter which, since they are all the same). There are then four possible ways to connect it to the top vertex. Now pick the next leg on the bottom vertex; there are three remaining ways to connect it to the top vertex. Repeating this procedure for the last two legs, we end up with a total of $4! = 24$ ways of connecting the diagram.

- The second diagram gives

 $$= \frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^2 (\mathbf{72}) \int \mathrm{d}^4x\,\mathrm{d}^4y\; \Delta_F^2(0)\Delta_F^2(x-y). \tag{7.7}$$

  The combinatorial factor of **72** arises as follows: There are ${}^4C_2 = 6$ ways of connecting two of the legs on the bottom vertex together, and there are also ${}^4C_2$ ways of connecting two of the legs on the top vertex together. Having done this, the bottom vertex has two remaining legs that need to be connected to the two remaining legs on the top vertex. There are two ways to do this. Overall, we get ${}^4C_2 \times {}^4C_2 \times 2 = 72$.

- The last diagram gives us

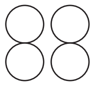
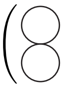
 $$= \frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^2 (\mathbf{9}) \int \mathrm{d}^4x\,\mathrm{d}^4y\; \Delta_F^4(0), \tag{7.8}$$

  where the combinatorial factor of **9** follows from the combinatorial factor of 3 of the $n = 1$ diagram. Since there are two copies, we get $3 \times 3 = 9$.

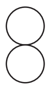
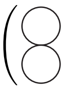
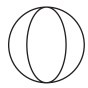
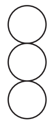
Adding it all up — If we stare at Eqs. (7.4) and (7.8) long enough, we recognize that

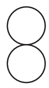
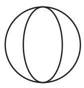
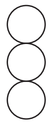
$$\frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^2 (\mathbf{9}) \int \mathrm{d}^4x\,\mathrm{d}^4y\; \Delta_F^4(0) = \frac{1}{2}\left[-i\frac{\lambda}{4!}(\mathbf{3}) \int \mathrm{d}^4x\; \Delta_F^2(0)\right]^2, \tag{7.9}$$

or diagrammatically,

 $$= \frac{1}{2}\Big($$  $$\Big)^2. \tag{7.10}$$

Thus, the vacuum-to-vacuum amplitude up to $\mathcal{O}(\lambda^2)$ is

$$\langle 0\,|S|\,0\rangle = 1 +$$  $$+ \frac{1}{2}\Big($$  $$\Big)^2 +$$  $$+$$  $$+ \mathcal{O}(\lambda^3)$$

$$= \exp\Big($$  $$+$$  $$+$$  $$+ \dots\Big). \tag{7.11}$$

> **Aside 7.1:** A good consistency check to do when counting combinatorial factors is to ensure that their sum adds up to
>
> $$P_n = \frac{(4n)!}{4^n(2n)!} \tag{7.12}$$
>
> at each level in n, i.e., at each order in λ. The number P_n is the total number of ways of joining $4n$ elements (the $4n$ factors of ϕ appearing in the correlation function) into distinct pairs. Let's check this for our calculations above: For $n = 1$, we have $P_1 = 3$ as expected. For $n = 2$, we get $P_2 = 105$, and we also have $24 + 72 + 9 = 105$.

Symmetry factors — If we simplify the numerical factors appearing in front of each diagram above, we get

$$\frac{\mathbf{3}}{4!} = \frac{1}{8}, \qquad \frac{\mathbf{24}}{2!(4!)^2} = \frac{1}{48}, \qquad \frac{\mathbf{72}}{2!(4!)^2} = \frac{1}{16}, \qquad \frac{\mathbf{9}}{2!(4!)^2} = \frac{1}{128}. \tag{7.13}$$

The numbers 8, 48, 16, and 128 appearing in the denominators are called the symmetry factors of the diagram. Rather than count combinatorial factors, we can also directly count symmetry factors. At low orders in λ, counting either combinatorial or symmetry factors both present about the same level of difficulty, but at higher orders, it becomes much easier to count symmetry factors.

To see where these numbers come from, consider what happens when we contract the $4n$-point correlation function $\langle 0| T\phi^4(x_1)\cdots\phi^4(x_n)|0\rangle$. Nothing changes if we connect something to the first factor of $\phi(x_1)$ rather than the second (or third, or fourth) factor of $\phi(x_1)$, so we would expect that every 4! contractions give the same result. In fact, this is true also for $\phi(x_2)$, $\phi(x_3)$, and so on, so really every $(4!)^n$ contractions give the same result. This cancels the $1/(4!)^n$ normalization coming from the vertex. Additionally, whether we call one vertex x_1 or x_2 or x_n should not matter, so there are also $n!$ contractions leading to the same result, cancelling the factor of $1/n!$ that came from expanding the exponential. It now seems that each diagram should have an overall numerical factor of 1, but we have overcounted at this point. Specifically, we have overcounted in five distinct ways, which we now need to correct for:[5]

(1) When a propagator starts and ends on the same vertex, we have overcounted by a factor of 2.

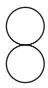
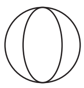
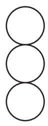
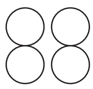
(2) If a pair of vertices is connected by k identical propagators, we have overcounted by a factor of $k!$.

(3) If vertices can be permuted without affecting the diagram, we have overcounted by the number of permutations.

(4) If a diagram contains n identical disconnected pieces, we have overcounted by a factor of $n!$.

(5) For each figure-8 ($n = 1$) bubble diagram, we overcount by an additional factor of 2.

The overall number by which we have overcounted is the symmetry factor S of the diagram. We correct for this overcounting by attaching a factor $1/S$ to each diagram. Notice that the value of $1/S$ for each diagram below is in agreement with the RHS's of Eq. (7.13).

	$S = 2^2 \times 2 = 8$	Two factors of 2 come from using Rule (1), and the last factor of 2 comes from Rule (5).
	$S = 4! \times 2 = 48$	The factor of 4! comes from Rule (2), whereas the factor of 2 comes from Rule (3).
	$S = 2^2 \times 2 \times 2 = 16$	Two factors of 2 come from Rule (1), one factor of 2 comes from Rule (2), and one last factor of 2 comes from Rule (3).
	$S = 8^2 \times 2 = 128$	Each factor of 8 is the symmetry factor for the $n = 1$ bubble diagram in isolation, and the factor of 2 comes from Rule (4).

Question 8

We consider an interacting theory involving three scalar fields ϕ_i, $i \in \{1, 2, 3\}$, governed by the Lagrangian

$$\mathcal{L} = \sum_{i=1}^{3}\left[\frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{2}m^2\phi_i^2\right] - \frac{\lambda}{8}\left(\sum_{i=1}^{3}\phi_i^2\right)^2 . \tag{8.1}$$

[5] See Chapter 4.4 of Peskin and Schroeder [2] or a paper by Dong *et al.* [3] for more words on this issue. For those with a strong mathematical predisposition, Sec. 2.3.1 of David Skinner's Advanced Quantum Field Theory lecture notes [4] provides a discussion on how to formalize these ideas with the concept of group orbits.

When $\lambda = 0$, this is a theory of three identical but independent scalar fields. Each scalar will have its own Feynman propagator,

$$\langle 0 | T\phi_i(x)\phi_j(x) | 0 \rangle = \begin{cases} \Delta_F(x-y) & i = j, \\ 0 & i \neq j. \end{cases} \tag{8.2}$$

Naturally, this can be written succinctly as

$$\langle 0 | T\phi_i(x)\phi_j(x) | 0 \rangle = \delta_{ij}\Delta_F(x-y), \tag{8.3}$$

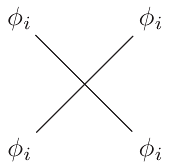
which gives the momentum-space Feynman rule

$$\phi_i \text{ ——————— } \phi_j = \frac{i\delta_{ij}}{p^2 - m^2}. \tag{8.4}$$

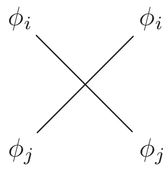
Let us now consider the interaction term. Writing out the sum explicitly gives

$$-\frac{\lambda}{8}\left(\sum_{i=1}^{3}\phi_i^2\right)^2 = -\frac{3\lambda}{4!}\left(\phi_1^4 + \phi_2^4 + \phi_3^4\right) - \frac{\lambda}{2!2!}\left(\phi_1^2\phi_2^2 + \phi_1^2\phi_3^2 + \phi_2^2\phi_3^2\right), \tag{8.5}$$

which allows us to read off the Feynman rules for two different types of vertices. The first vertex involves the interaction between four powers of the same field, while the second vertex involves interactions between two powers each of two different fields:

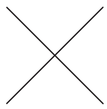


$= -3i\lambda,$



$= -i\lambda \quad (i \neq j).$ (8.6)

Instead of treating them separately, it is possible to combine both vertices by writing



$$= -i\lambda(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}). \tag{8.7}$$

This last expression is equal to the amplitude[6] $i\mathcal{M}$ for $\phi_i\phi_j \to \phi_k\phi_\ell$ scattering at lowest, nontrivial order in λ.

Question 9

Consider the theory

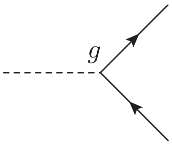
$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi - m^2\psi^*\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 - g\psi^*\psi\phi - h|\psi|^4 - k\phi^3 - \ell\partial_\mu\psi\partial^\mu\psi^*\phi. \tag{9.1}$$

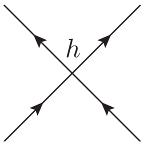
The momentum-space Feynman rules are as follows: The propagators for the complex scalar ψ and the real scalar ϕ are, respectively,

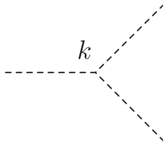
$$\xrightarrow{\quad p \quad} = \frac{i}{p^2 - m^2}, \qquad \text{- - - - - - - -} = \frac{i}{p^2 - M^2}.$$

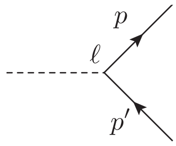
Notice that the arrow on the propagator for the complex field serves two purposes: it denotes the direction of momentum flow, and distinguishes between ψ on one end and ψ^* on the other. The four interaction vertices give:

[6]Note some sources write $\mathcal{A}$ in place of $\mathcal{M}$.



$= -ig,$


$= -4ih,$


$= -6ik,$


$= -i\ell(p \cdot p').$

The factor of $4 = 2!\,2!$ in the h vertex comes from the fact that there are 2! ways to contract the factors of $\psi(x)$ with the rest of the Feynman diagram, and likewise there are also 2! ways to contract the factors of $\psi^*(x)$. Similarly, the factor of $6 = 3!$ in the k vertex is due to there being 3! ways to contract $\phi(x)$.

Since the Lagrangian must have mass dimension $[\mathcal{L}] = 4$ (in a four-dimensional universe) while $[\phi] = [\psi] = [\psi^*] = [\partial_\mu] = +1$, it follows that

$$[g] = [k] = +1, \quad [h] = 0, \quad [\ell] = -1. \tag{9.2}$$

The mass dimension of ℓ tells us that its associated term in the Lagrangian is an irrelevant operator. The theory is said to be "non-renormalizable."

Question 10

We consider the same theory as in Question 9, except with $h = k = \ell = 0$. The real scalar field ϕ will be referred to as the "meson," whereas the complex scalar ψ will be called the "nucleon."

(a) From Eq. (18) of Ben Allanach's notes [5], the decay of a particle of mass M in its rest frame into n particles with 4-momenta q_i proceeds at a rate

$$\Gamma = \frac{1}{2M} \int \prod_{i=1}^{n} \frac{\mathrm{d}^3 q_i}{(2\pi)^3 2E_{q_i}} |\mathcal{M}|^2 (2\pi)^4 \delta^{(4)}\left(p - \textstyle\sum_{i=1}^n q_i\right). \tag{10.1}$$

Proposition 10.1: *If $M > 2m$, meson decay $\phi \to \psi\psi^\dagger$ at leading order in g proceeds with a decay width*

$$\Gamma = \frac{g^2}{16\pi M}\sqrt{1 - \left(\frac{2m}{M}\right)^2}. \tag{10.2}$$

Proof.—The Feynman rules we established in Question 9 tell us that meson decay has a probability amplitude $i\mathcal{M} = -ig + \mathcal{O}(g^2)$. Plug this into the master formula in Eq. (10.1) with $n = 2$ to get

$$\Gamma = \frac{g^2}{2M} \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3 q_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}(p - q_1 - q_2), \tag{10.3}$$

writing $E_i \equiv E_{q_i}$ for simplicity. In its rest frame, the meson has 4-momentum $p^\mu = (M, \mathbf{0})$, thus the delta function factorizes as $\delta^{(4)}(p - q_1 - q_2) = \delta(E_1 + E_2 - M)\delta^{(3)}(\mathbf{q}_1 - \mathbf{q}_2)$. We integrate over $\mathbf{q}_2$ to get

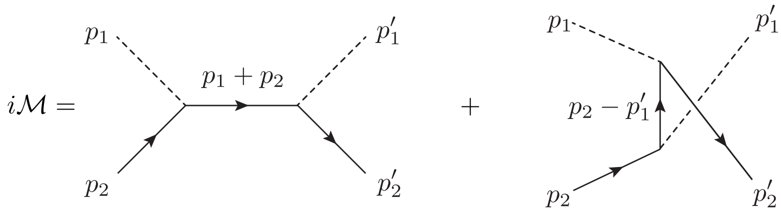
$$\Gamma = \frac{g^2}{2M} \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3 4E_1^2} (2\pi)\delta(2E_1 - M) = \frac{g^2}{2\pi M} \int_0^\infty \frac{q^2 dq}{(2E)^2} \delta(2E - M), \tag{10.4}$$

having moved into spherical coordinates in the last step, while also dropping the subscript "1" that has become superfluous. Rather than integrate over $q \equiv |\mathbf{q}|$, it is most convenient to change variables and integrate over $x = 2E$. The chain rule tells us that $E\ \mathrm{d}E = q\ \mathrm{d}q$, thus

$$\Gamma = \frac{g^2}{2\pi M} \int_m^\infty \frac{E\sqrt{E^2 - m^2}\,\mathrm{d}E}{(2E)^2} \delta(2E - M) = \frac{g^2}{2\pi M} \int_{2m}^\infty \frac{\sqrt{(x/2)^2 - m^2}\,\mathrm{d}x}{4x} \delta(x - M). \tag{10.5}$$

Since $M > 2m$, the integral over x returns a nonzero value which produces the desired result. ∎

(b) We now consider nucleon–meson scattering, $\phi\psi \to \phi\psi$, which has an amplitude given at tree level by the sum of two diagrams,



$$= (-ig)^2 \left[\frac{i}{(p_1+p_2)^2 - m^2} + \frac{i}{(p_2 - p_1')^2 - m^2} \right]. \tag{10.6}$$

Proposition 10.2: *The differential scattering cross section for nucleon–meson scattering in the centre-of-mass frame is*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{g^4 (s+u-2m^2)^2}{16\pi (s-m^2)^2 (u-m^2)^2 (s^2 + m^4 + M^4 - 2sm^2 - 2sM^2 - 2m^2M^2)}, \tag{10.7}$$

where $s = (p_1+p_2)^2$, $t = (p_1 - p_1')^2$, and $u = (p_2 - p_1')^2$ are the usual Mandelstam variables.

Proof.—From Eq. (16) of Ben Allanach's notes [5], any $2 \to 2$ process has the differential scattering cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{|\mathcal{M}|^2}{16\pi\lambda(s, m_1^2, m_2^2)} \tag{10.8}$$

in its centre-of-mass frame, with $\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$. Note m_1 and m_2 are the masses of the incoming particles. Setting $m_1 = m$, $m_2 = M$, and using the expression for $\mathcal{M}$ in Eq. (10.6), we recover the desired result. ■

Of course, the three Mandelstam variables are not all independent, and are related by the identity

$$s + t + u = \sum_i m_i^2 = 2(m^2 + M^2). \tag{10.9}$$

This equation can be used, for instance, to rewrite $\mathrm{d}\sigma/\mathrm{d}t$ as a function of (s,t) rather than (s,u).

It is easy to check that Eq. (10.7) is dimensionally consistent. The scattering cross section has units of area, thus $[\sigma] = -2$, whereas $[t] = +2$, thus the LHS of Eq. (10.7) has mass dimension -4. The RHS also has mass dimension -4, since $[g] = [m] = [M] = +1$ and $[s] = [u] = +2$.

References

[1] D. Tong, *Quantum Field Theory*, (2007) http://www.damtp.cam.ac.uk/user/tong/qft.html.

[2] M. E. Peskin and D. V. Schroeder, *An Introduction To Quantum Field Theory* (Perseus Books Publishing, 1995).

[3] P. V. Dong, L. T. Hue, H. T. Hung, H. N. Long, and N. H. Thao, "Symmetry Factors of Feynman Diagrams for Scalar Fields", Theor. Math. Phys. **165**, 1500 (2010), arXiv:0907.0859.

[4] D. Skinner, *Quantum Field Theory II*, (2018) http://www.damtp.cam.ac.uk/user/dbs26/AQFT/chap2.pdf.

[5] B. C. Allanach, *QFT: Cross Sections and Decay Rates*, (2018) http://www.damtp.cam.ac.uk/user/examples/3P1l.pdf.