Quantum Field Theory: Example Sheet 4

Michaelmas 2018, Mathematical Tripos Part III, University of Cambridge

Model solutions by L. K. Wong*. Last updated January 7, 2019.

Question 1

We consider Yukawa theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\not{\partial} - m)\psi - \lambda\phi\bar{\psi}\psi. \tag{1.1}$$

Let $s = (p+q)^2$, $t = (p-p')^2$, and $u = (p-q')^2$ be the usual Mandelstam variables.

Proposition 1.1: *Nucleon–antinucleon scattering, $\psi\bar{\psi} \to \psi\bar{\psi}$, has the tree-level amplitude*

$$\mathcal{A} = (-i\lambda)^2 \left(\frac{[\bar{v}^r(\mathbf{q}) \cdot u^s(\mathbf{p})][\bar{u}^{s'}(\mathbf{p}') \cdot v^{r'}(\mathbf{q}')]}{s - \mu^2} - \frac{[\bar{u}^{s'}(\mathbf{p}') \cdot u^s(\mathbf{p})][\bar{v}^r(\mathbf{q}) \cdot v^{r'}(\mathbf{q}')]}{t - \mu^2} \right). \tag{1.2}$$

Proof.—Obtaining this result using Feynman rules is straightforward; here, we will take the long and hard route to see how it emerges from Dyson's formula and the canonical formalism. We want to compute the S-matrix element

$$\begin{aligned} \langle f | \, S \, | i \rangle &= \langle f | \, T \exp\left(-i\lambda \int \mathrm{d}^4x \; \phi\bar{\psi}\psi \right) | i \rangle \\ &= \langle f \, | i \rangle + (-i\lambda) \int \mathrm{d}^4x \, \langle f | \, T[\phi\bar{\psi}\psi] \, | i \rangle + \mathcal{O}(\lambda^2) \end{aligned} \tag{1.3}$$

for initial and final states given by

$$|i\rangle = \sqrt{4E_p E_q} b_p^{s\dagger} c_q^{r\dagger} |0\rangle, \quad |f\rangle = \sqrt{4E_p E_q} b_{p'}^{s'\dagger} c_{q'}^{r'\dagger} |0\rangle. \tag{1.4}$$

We are never interested in the noninteracting $\mathcal{O}(\lambda^0)$ part of the S matrix, while the first-order term vanishes since the single factor of $\phi(x)$ will annihilate both $|i\rangle$ and $\langle f|$. Thus, the leading nontrivial part of this S-matrix element is quadratic in λ:

$$\langle f | \, S \, | i \rangle \supset \frac{1}{2}(-i\lambda)^2 \int \mathrm{d}^4x \, \mathrm{d}^4y \, \langle f | \, T[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)] \, | i \rangle . \tag{1.5}$$

At this stage, we use Wick's theorem to rewrite the time-ordered product in terms of Wick contractions and normal-ordered products. In doing so, we obtain several types of terms that are irrelevant:

- Terms in which $\phi(x)$ is *not* Wick contracted with $\phi(y)$ will annihilate $|i\rangle$, thus will not contribute.
- Terms in which any of the spinors are Wick contracted with one another lead to disconnected diagrams, which should be neglected.[1]

It follows that the only physical contribution comes from Wick contracting $\phi(x)$ with $\phi(y)$,

$$T[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)] \supset \underbrace{\phi(x)\phi(y)}_{\Delta_F(x-y)} \; :[\bar{\psi}(x)\psi(x)][\bar{\psi}(y)\psi(y)]: . \tag{1.6}$$

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[1] It will not be clear from our approach why disconnected diagrams ought to be neglected. The interested reader should consult classic texts on the LSZ reduction formula, such as Chapter 7.2 of Peskin and Schroeder [1].

Recall Δ_F is the Feynman propagator for the scalar field. Note also that the spinor index on $\bar{\psi}(x)$ is always contracted with $\psi(x)$, and likewise for $\bar{\psi}(y)$ and $\psi(y)$; this fact is often emphasized by placing square brackets around each spinor product. Now use its Fourier decomposition to write

$$\psi(x) = \sum_s \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(b_p^s u^s(\mathbf{p}) e^{-ip\cdot x} + c_p^{s\dagger} v^s(\mathbf{p}) e^{ip\cdot x} \right). \tag{1.7}$$

Schematically, this is $\psi \sim b + c^\dagger$. Using this expansion, only terms of the form $:\bar{\psi}\psi\bar{\psi}\psi: \sim b^\dagger c^\dagger c b$ will contribute, since the initial and final states are $|i\rangle \sim b^\dagger c^\dagger |0\rangle$ and $\langle f| \sim \langle 0| c\, b$; all others will annihilate the vacuum. There are four terms of this form:

$$\begin{aligned} :[\bar{\psi}(x)\psi(x)][\bar{\psi}(y)\psi(y)]: \supset \sum_{\text{spins}} \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3} \cdots \frac{\mathrm{d}^3 p_4}{(2\pi)^3} \frac{1}{\sqrt{16E_1E_2E_3E_4}} \\ \times \Big\{ :b_1^\dagger c_2^\dagger c_3 b_4: [\bar{u}_1 \cdot v_2][\bar{v}_3 \cdot u_4] e^{i(+p_1\cdot x + p_2 \cdot x - p_3 \cdot y - p_4 \cdot y)} \\ + :c_1 b_2 b_3^\dagger c_4^\dagger: [\bar{v}_1 \cdot u_2][\bar{u}_3 \cdot v_4] e^{i(-p_1\cdot x - p_2 \cdot x + p_3 \cdot y + p_4 \cdot y)} \\ + :b_1^\dagger b_2 c_3 c_4^\dagger: [\bar{u}_1 \cdot u_2][\bar{v}_3 \cdot v_4] e^{i(+p_1\cdot x - p_2 \cdot x + p_3 \cdot y - p_4 \cdot y)} \\ + :c_1 c_2^\dagger b_3^\dagger b_4: [\bar{v}_1 \cdot v_2][\bar{u}_3 \cdot u_4] e^{i(-p_1\cdot x + p_2 \cdot x + p_3 \cdot y - p_4 \cdot y)} \Big\}. \end{aligned} \tag{1.8}$$

Above, I have written $u_i \equiv u^{s_i}(\mathbf{p}_i)$ and $E_i \equiv E_{p_i}$ as shorthand. Through judicious relabeling of the dummy indices and integration variables, one can check that the first and second lines in curly brackets are identical, and similarly the third and fourth lines are identical. Hence, the above equation simplifies to yield

$$\begin{aligned} 2\sum_{\text{spins}} \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3} \cdots \frac{\mathrm{d}^3 p_4}{(2\pi)^3} \frac{1}{\sqrt{16E_1E_2E_3E_4}} \Big\{ b_1^\dagger c_2^\dagger c_3 b_4 [\bar{u}_1 \cdot v_2][\bar{v}_3 \cdot u_4] e^{i(+p_1\cdot x + p_2 \cdot x - p_3 \cdot y - p_4 \cdot y)} \\ - b_1^\dagger c_4^\dagger c_3 b_2 [\bar{u}_1 \cdot u_2][\bar{v}_3 \cdot v_4] e^{i(+p_1\cdot x - p_2 \cdot x + p_3 \cdot y - p_4 \cdot y)} \Big\}. \end{aligned} \tag{1.9}$$

The minus sign in front of the second term arises because all fermionic operators *anti*commute with one another under normal ordering. The overall factor of 2 will cancel the factor of 1/2 in Eq. (1.5) that came from expanding the exponential to second order.

Putting everything together, we get

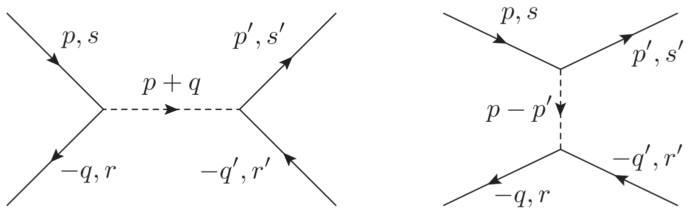
$$\begin{aligned} \langle f| S |i\rangle &\supset \frac{1}{2}(-i\lambda)^2 \int \mathrm{d}^4x\, \mathrm{d}^4y\ \Delta_F(x-y) \langle f| :[\bar{\psi}(x)\psi(x)][\bar{\psi}(y)\psi(y)]: |i\rangle \\ &= (-i\lambda)^2 \int \mathrm{d}^4x\, \mathrm{d}^4y \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{ie^{-ik\cdot(x-y)}}{k^2-\mu^2} \sum_{\text{spins}} \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3} \cdots \frac{\mathrm{d}^3 p_4}{(2\pi)^3} \langle 0| c_{q'}^{r'} b_{p'}^{s'} \{\cdots\} b_p^{s\dagger} c_q^{r\dagger} |0\rangle. \end{aligned} \tag{1.10}$$

In obtaining the second line, we neglected to write down factors of $\sqrt{E}$ since they will eventually cancel each other out, and we write $\{\cdots\}$ to mean everything in the curly brackets of Eq. (1.9). Standard anticommutation relations can be used to show that the first term in $\{\cdots\}$ is proportional to

$$\begin{aligned} \langle 0| c_{q'}^{r'} b_{p'}^{s'} b_1^\dagger c_2^\dagger c_3 b_4 b_p^{s\dagger} c_q^{r\dagger} |0\rangle = (2\pi)^{12} \delta^{s_1,s'} \delta^{s_2,r'} \delta^{s_3,r} \delta^{s_4,s} \\ \times \delta^{(3)}(\mathbf{p}_1 - \mathbf{p}') \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}') \delta^{(3)}(\mathbf{p}_3 - \mathbf{q}) \delta^{(3)}(\mathbf{p}_4 - \mathbf{p}). \end{aligned} \tag{1.11}$$

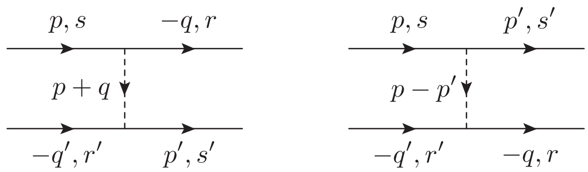
The second term is proportional to a similar product of delta functions up to the interchange $2 \leftrightarrow 4$. Evaluating all the integrals, we find that the S-matrix element takes the form $i\mathcal{A}\,(2\pi)^4 \delta^{(4)}(p'+q'-p-q)$, where $\mathcal{A}$ is the desired amplitude. ■

This amplitude corresponds to the following two Feynman diagrams:



Using the Feynman rules in the Appendix for the propagators and external legs, and multiplying by a factor of $-i\lambda$ for each interaction vertex, one easily recovers the result in Eq. (1.2). The one subtlety is getting the relative minus sign correct. We understand from having done the calculation explicitly above that it comes from anticommuting the creation and annihilation operators past one another, but how can we tell that there should be a minus sign when just looking at the diagrams? The general rule is as follows:

(1) Draw each diagram such that all fermion lines are horizontal, with their arrows all pointing from left to right,[2] as shown below:



(2) Label each left external fermion leg in the same fixed order (from top to bottom). In this example, the momentum p is always on top, while the momentum $-q'$ is always at the bottom.

(3) Pick any arbitrary Feynman diagram as a point of reference.[3]

(4) For the remaining diagrams, observe the ordering (from top to bottom) of the momenta on the right external fermion legs. If this ordering is obtained through an even (odd) permutation of the momenta in the reference diagram, then we pick up a plus (minus) sign.[4]

Proposition 1.2: *Nucleon–meson scattering, $\psi\phi \to \psi\phi$, has the tree-level amplitude*

$$\mathcal{A} = (-i\lambda)^2 \left(\frac{[\bar{u}^{s'}(\mathbf{p}')(\not{p} + \not{q} + m)u^s(\mathbf{p})]}{s - m^2} + \frac{[\bar{u}^{s'}(\mathbf{p}')(\not{p} - \not{q}' + m)u^s(\mathbf{p})]}{u - m^2} \right). \tag{1.12}$$

Proof.—It will be instructive to also derive this result the long and hard way. Our initial and final states are[5]

$$|i\rangle = \sqrt{4E_p E_q} a_q^\dagger b_p^{s\dagger} |0\rangle \,, \quad |f\rangle = \sqrt{4E_{p'} E_{q'}} a_{q'}^\dagger b_{p'}^{s'\dagger} |0\rangle \,, \tag{1.13}$$

and the relevant S-matrix element is

$$\langle f| S |i\rangle \supset \frac{1}{2}(-i\lambda)^2 \int \mathrm{d}^4x \, \mathrm{d}^4y \, \langle f| T[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)] |i\rangle \tag{1.14}$$

at leading order. As before, we ignore the noninteracting part of the S matrix at $\mathcal{O}(\lambda^0)$, and note that the $\mathcal{O}(\lambda)$ term evaluates to zero. Using Wick's theorem and discarding all disconnected diagrams, the

[2] This sometimes means that we have both initial and final-state particles on either side of the diagram. This isn't a problem; we're drawing diagrams in this way only to figure out their relative signs.
[3] Depending on which diagram you pick, the total amplitude $\mathcal{A}$ may be off by an overall minus sign. This isn't a problem, since physical observables only depend on $|\mathcal{A}|^2$.
[4] See, e.g., Chapter 45 of Srednicki [2] for more details.
[5] One understands from context which mass, m or μ, to put inside the definition of E_p.

only contribution comes from Wick contracting one of the spinors at x with one of the spinors at y. Since $\overbracket{\psi\psi} = 0$ and $\overbracket{\bar{\psi}\bar{\psi}} = 0$, there are two terms that contribute:

$$
\begin{aligned}
T[\phi(x)\bar{\psi}_\alpha(x)\psi^\alpha(x)\phi(y)\bar{\psi}_\beta(y)\psi^\beta(y)] \supset\ & \overbracket{\psi^\alpha(x)\bar{\psi}_\beta(y)}\ {:}\phi(x)\bar{\psi}_\alpha(x)\phi(y)\psi^\beta(y){:} \\
& + \overbracket{\psi^\beta(y)\bar{\psi}_\alpha(x)}\ {:}\phi(x)\bar{\psi}_\beta(y)\phi(y)\psi^\alpha(x){:}\ .
\end{aligned}
\tag{1.15}
$$

Keep in mind that fermionic operators anticommute past each other inside the time and normal-ordered products. The spinor indices α, β are being summed over, and the position variables x, y are being integrated over, so they can be judiciously renamed. Noting that ϕ freely commutes with the fermion, it follows that both terms yield identical contributions, hence the rhs of Eq. (1.15) can be replaced by

$$
2(S_F)^\alpha{}_\beta(x-y)\ {:}\phi(x)\bar{\psi}_\alpha(x)\phi(y)\psi^\beta(y){:}, \tag{1.16}
$$

where $S_F = \overbracket{\psi\bar{\psi}}$ is the Feynman propagator for the fermion. The overall factor of 2 here will eventually cancel the factor of 1/2 in Eq. (1.14) that came from expanding the exponential to second order.

Let us now turn to the remaining normal-ordered product. Schematically, the initial and final states are $|i\rangle \sim a^\dagger b^\dagger |0\rangle$ and $\langle f| \sim \langle 0| ba$, so only terms of the form ${:}\phi\bar{\psi}\phi\psi{:} \sim a^\dagger b^\dagger ab$ will contribute. There are two such terms:

$$
\begin{aligned}
{:}\phi(x)\bar{\psi}_\alpha(x)\phi(y)\psi^\beta(y){:} \supset \sum_{\text{spins}} \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3} \cdots \frac{\mathrm{d}^3 p_4}{(2\pi)^3} \frac{1}{\sqrt{16E_1E_2E_3E_4}} (\bar{u}_2)_\alpha (u_4)^\beta \\
\times \Big\{ {:}a_1^\dagger b_2^\dagger a_3 b_4{:}\ e^{i(+p_1\cdot x + p_2\cdot x - p_3\cdot y - p_4\cdot y)} \\
+ {:}a_1 b_2^\dagger a_3^\dagger b_4{:}\ e^{i(-p_1\cdot x + p_2\cdot x + p_3\cdot y - p_4\cdot y)} \Big\},
\end{aligned}
\tag{1.17}
$$

using the shorthand notation introduced earlier. Putting everything together and paying close attention to how the spinor indices are being contracted, we find

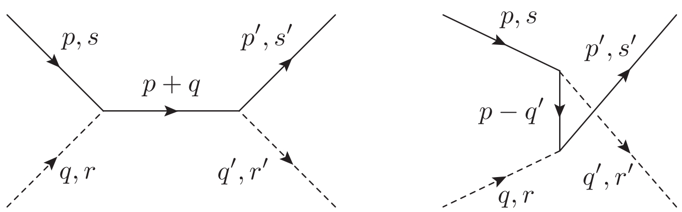
$$
\begin{aligned}
\langle f| S |i\rangle \supset (-i\lambda)^2 \int \mathrm{d}^4x\, \mathrm{d}^4y \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{i[\bar{u}_2(\not{k} + m)u_4]}{k^2 - m^2} e^{-ik\cdot(x-y)} \\
\times \sum_{\text{spins}} \int \frac{\mathrm{d}^3p_1}{(2\pi)^3} \cdots \int \frac{\mathrm{d}^3p_4}{(2\pi)^3} \langle 0| b_{p'}^{s'} a_{q'} \{\cdots\} a_q^\dagger b_p^{s\dagger} |0\rangle\, .
\end{aligned}
\tag{1.18}
$$

As we did earlier, factors of $\sqrt{E}$ have been ignored since they will eventually cancel each other out. Standard (anti)commutation rules can be used to show that

$$
\langle 0| b_{p'}^{s'} a_{q'} a_1^\dagger b_2^\dagger a_3 b_4 a_q^\dagger b_p^{s\dagger} |0\rangle = (2\pi)^{12} \delta^{s_2,s'} \delta^{s_4,s} \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}')\delta^{(3)}(\mathbf{p}_2 - \mathbf{p}')\delta^{(3)}(\mathbf{p}_3 - \mathbf{q})\delta^{(3)}(\mathbf{p}_4 - \mathbf{p}). \tag{1.19}
$$

A similar product of delta functions is obtained for the second term in $\{\cdots\}$. Finally, evaluating all the integrals allows us to read off the desired result. ■

The Feynman diagrams for this process are:



Question 2

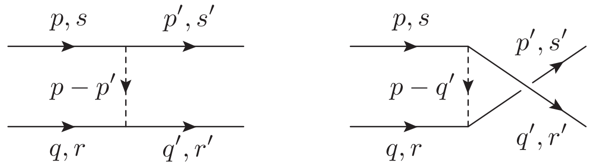
Consider pseudoscalar Yukawa theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - \lambda\phi\bar{\psi}\gamma^5\psi. \tag{2.1}$$

The Feynman rules for the propagators and external legs are always the same, and are collated in the Appendix at the end. For this theory, each interaction vertex gives a factor $-i\lambda\gamma^5$. These rules can be used to show that $\psi\psi \to \psi\psi$ scattering has the tree-level amplitude

$$\mathcal{A}(\psi\psi \to \psi\psi) = (-i\lambda)^2 \left(\frac{[\bar{u}^{s'}(\mathbf{p}')\gamma^5 u^s(\mathbf{p})][\bar{u}^{r'}(\mathbf{q}')\gamma^5 u^r(\mathbf{q})]}{t-\mu^2} - \frac{[\bar{u}^{s'}(\mathbf{p}')\gamma^5 u^r(\mathbf{q})][\bar{u}^{r'}(\mathbf{q}')\gamma^5 u^s(\mathbf{p})]}{u-\mu^2} \right), \tag{2.2}$$

corresponding to the following two diagrams:



Similarly, $\psi\bar{\psi} \to \psi\bar{\psi}$ scattering has the amplitude

$$\mathcal{A}(\psi\bar{\psi} \to \psi\bar{\psi}) = (-i\lambda)^2 \left(\frac{[\bar{v}^{r}(\mathbf{q})\gamma^5 u^s(\mathbf{p})][\bar{u}^{s'}(\mathbf{p}')\gamma^5 v^{r'}(\mathbf{q}')]}{s-\mu^2} - \frac{[\bar{u}^{s'}(\mathbf{p}')\gamma^5 u^s(\mathbf{p})][\bar{v}^{r}(\mathbf{q})\gamma^5 v^{r'}(\mathbf{q}')]}{t-\mu^2} \right). \tag{2.3}$$

The corresponding diagrams are equivalent to those presented in Question 1.

Question 3

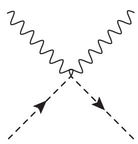
Scalar QED has the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D^\mu\phi)^\dagger(D_\mu\phi) - \mu^2\phi^\dagger\phi, \tag{3.1}$$

where ϕ is a complex scalar and $D_\mu = \partial_\mu + ieA_\mu$ is the gauge-covariant derivative. We can rewrite this Lagrangian as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$, where terms for the free Maxwell field and the free scalar are encapsulated in $\mathcal{L}_0$, while interactions between the two are given by

$$\mathcal{L}_{\text{int}} = -ieA_\mu(\phi^\dagger\partial^\mu\phi - \phi\,\partial^\mu\phi^\dagger) + e^2A_\mu A^\mu\phi^\dagger\phi. \tag{3.2}$$

The two terms linear in A_μ will be treated as one interaction vertex, and the remaining term quadratic in A_μ should be treated as a separate vertex. The Feynman rule for the latter is easy to read off;

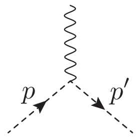


$$= 2ie^2\eta_{\mu\nu}. \tag{3.3}$$

Note the factor of 2, which arises because there are two indistinguishable copies of A_μ in the vertex.

Because of the derivative, it is not as easy to read off the Feynman rule for the three-point vertex; especially if we want to get minus signs and factors of i right. Let's see how to derive it systematically.

Proposition 3.1: *The Feynman rule for the three-point vertex in scalar QED is*



$$= -ie(p+p')_\mu. \tag{3.4}$$

Proof.—We will prove this by considering a suitable process and computing its amplitude using Dyson's formula. Suppose an ingoing scalar with momentum p spontaneously absorbs a photon of momentum q, resulting in an outgoing scalar with momentum p'. The initial and final states are

$$|i\rangle = \sqrt{4E_p E_q} b_p^\dagger a_q^{\lambda\dagger} |0\rangle, \quad |f\rangle = \sqrt{2E_{p'}} b_{p'}^\dagger |0\rangle. \tag{3.5}$$

Note that (b, c) are the annihilation operators for the scalar and its antiparticle, while the operator a is associated with the photon. The photon's polarization states are labeled by the index λ.

The relevant S-matrix element is

$$\langle f| S |i\rangle = i(-ie) \int \mathrm{d}^4x \, \langle f| T[A_\mu \phi^\dagger \partial^\mu \phi - A_\mu \phi \partial^\mu \phi^\dagger] |i\rangle + \mathcal{O}(e^2). \tag{3.6}$$

Unlike in Question 1, there is no zeroth-order term since the initial and final states don't overlap. We proceed by using Wick's theorem to expand the time-ordered product. The only term that contributes is the one in which there are no Wick contractions; the rest correspond to unphysical, disconnected diagrams. Thus

$$\langle f| S |i\rangle = e \int \mathrm{d}^4x \, \langle f| {:}A_\mu \phi^\dagger \partial^\mu \phi - A_\mu \phi \partial^\mu \phi^\dagger{:} |i\rangle. \tag{3.7}$$

As $|i\rangle \sim b^\dagger a^\dagger |0\rangle$ while $\langle f| \sim \langle 0| b$, the only terms that contribute in the normal-ordered product are those of the form $\sim b^\dagger b \, a$. Substituting in their Fourier decompositions,

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(b_p e^{-ip\cdot x} + c_p^\dagger e^{ip\cdot x} \right), \tag{3.8}$$

$$A_\mu(x) = \sum_\lambda \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(\mathcal{E}_\mu^{\lambda *}(\mathbf{p}) \, a_p^\lambda e^{-ip\cdot x} + \mathcal{E}_\mu^\lambda(\mathbf{p}) \, a_p^{\lambda\dagger} e^{ip\cdot x} \right), \tag{3.9}$$

the part that contributes is

$$\begin{aligned} {:}A_\mu \phi^\dagger \partial^\mu \phi - A_\mu \phi \partial^\mu \phi^\dagger{:} \supset \sum_{\lambda'} \int \frac{\mathrm{d}^3p_1}{(2\pi)^3} \cdots \frac{\mathrm{d}^3p_3}{(2\pi)^3} \frac{1}{\sqrt{8E_1E_2E_3}} \mathcal{E}_\mu^{\lambda' *}(\mathbf{p}_1) \\ \times \left\{ {:}a_1^{\lambda'} b_2^\dagger b_3{:} \, (-ip_3^\mu) e^{i(-p_1+p_2-p_3)\cdot x} - {:}a_1^{\lambda'} b_2 b_3^\dagger{:} \, (+ip_3^\mu) e^{i(-p_1-p_2+p_3)\cdot x} \right\}. \end{aligned} \tag{3.10}$$

Putting everything together, we find

$$\langle f| S |i\rangle = e \int \mathrm{d}^4x \sum_{\lambda'} \int \frac{\mathrm{d}^3p_1}{(2\pi)^3} \cdots \frac{\mathrm{d}^3p_3}{(2\pi)^3} \mathcal{E}_\mu^{\lambda' *}(\mathbf{p}_1) \, \langle 0| b_{p'} \{ \cdots \} \, b_p^\dagger a_q^{\lambda\dagger} |0\rangle, \tag{3.11}$$

having as usual omitted any factors of $\sqrt{E}$ since they will eventually cancel out. The first term in $\{\cdots\}$ is proportional to

$$\langle 0| b_{p'} b_2^\dagger b_3 a_1^{\lambda'} b_p^\dagger a_q^{\lambda\dagger} |0\rangle = (2\pi)^9 \delta^{\lambda\lambda'} \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}) \delta^{(3)}(\mathbf{p}_2 - \mathbf{p}') \delta^{(3)}(\mathbf{p}_3 - \mathbf{p}). \tag{3.12}$$

The second term is proportional to a similar product of delta functions up to the interchange $2 \leftrightarrow 3$.

Evaluating all the integrals, one finds

$$\langle f|\, S\,|i\rangle = -ie(p+p')^\mu \; \mathcal{E}_\mu^{\lambda *}(\mathbf{q}) \; (2\pi)^4 \delta^{(4)}(p+q-p'). \tag{3.13}$$

The factor of $\mathcal{E}_\mu^{\lambda *}$ is related to the external photon leg, while the delta function at the end enforces overall momentum conservation. What remains is the contribution of the interaction vertex, so we deduce that the Feynman rule is, indeed, $-ie(p+p')^\mu$. ■

One might be slightly skeptical of this result, since the process we have considered is kinematically forbidden—it is impossible for a charged particle to spontaneously absorb a photon, as the process will not conserve both energy and 3-momentum simultaneously. Mathematically, this is captured by the delta function in Eq. (3.13), which will always evaluate to zero for all on-shell values of the external momenta. This isn't a problem. Whether a process is *kinematically* allowed or forbidden by momentum conservation has no bearing on the *dynamical* content of the theory—the interaction vertex will behave all the same, so it is indeed valid to use this process to derive the Feynman rules.

Aside 3.1: The first-principled approach above is a sure-fire way to correctly derive Feynman rules, but it is clearly not the most efficient. Having understood how to do things "properly," let us discuss a neat trick that gives us the same result. Just by inspection, we know that the three-point vertex must yield something like $-ie(p+p')^\mu$; each factor of the momentum coming from Fourier transforming the derivatives ∂^μ in the Lagrangian. All we really have to figure out is the overall numerical factor out front (including any factors of i), and the relative sign between p and p'.

To do this, let us focus on the part of the action that reads

$$S \supset -ie \int \mathrm{d}^4x \; A_\mu(\phi^\dagger \partial^\mu \phi - \phi\, \partial^\mu \phi^\dagger). \tag{3.14}$$

We can move into momentum space already at this stage. Use the Fourier transform to write

$$A_\mu(x) = \int \frac{\mathrm{d}^4q}{(2\pi)^4} e^{-iq\cdot x} \tilde{A}_\mu(q), \quad \phi(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-ip\cdot x} \tilde{\phi}(p), \quad \phi^\dagger(x) = \int \frac{\mathrm{d}^4p'}{(2\pi)^4} e^{-ip'\cdot x} \tilde{\phi}^\dagger(p'). \tag{3.15}$$

Notice that we are treating ϕ and $\phi^\dagger$ independently, which is why we choose to write $e^{-ip'\cdot x}$ rather than $e^{+ip'\cdot x}$ in the Fourier transform of $\phi^\dagger$. This choice will lead to additional insight later. Substitute these expressions back into the action to get

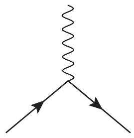
$$\begin{aligned} S &\supset -ie \int \mathrm{d}^4x \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\mathrm{d}^4p'}{(2\pi)^4} \frac{\mathrm{d}^4q}{(2\pi)^4} (-i)(p-p')^\mu \tilde{A}^\mu(q) \tilde{\phi}(p) \tilde{\phi}^\dagger(p') e^{-i(p+p'+q)\cdot x} \\ &= -e \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\mathrm{d}^4p'}{(2\pi)^4} \frac{\mathrm{d}^4q}{(2\pi)^4} (p-p')^\mu \tilde{A}^\mu(q) \tilde{\phi}(p) \tilde{\phi}^\dagger(p') \; (2\pi)^4 \delta^{(4)}(p+p'+q). \end{aligned} \tag{3.16}$$

Now we can read off the Feynman rule with the same ease as with non-derivative interaction terms. We ignore the integrals over the momenta and also ignore the fields. In addition, we also ignore the delta function and its accompanying factor of $(2\pi)^4$, which is enforcing momentum conservation at this vertex. What remains is $-ie(p-p')^\mu$; where the extra factor of i comes from the factor of i in Dyson's formula, as usual.

This is *almost* the right result, but the sign in front of p' is wrong. Why? This goes back to Eq. (3.15), and our choice to consistently put a minus sign inside the exponent for all three Fourier transforms. Physically, this corresponds to defining the vertex such that all three momenta (p, p', q) are flowing *into* the vertex. However, our convention is that p' is flowing *out*, and so all we have to do is flip the sign of p' to get the right result.

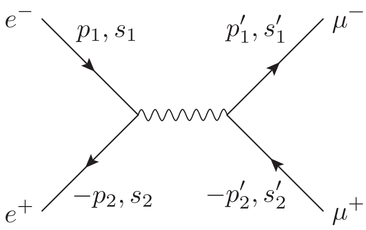
Question 4

Both the electron and muon are spin-half fermions with charge $-e$ that couple to the photon via the interaction vertex



$$= -ie\gamma^{\mu}. \tag{4.1}$$

Together with the Feynman rules for propagators and external legs given in the Appendix, one can show that the process $e^+e^- \to \mu^+\mu^-$ has amplitude $\mathcal{A}$ given by



$$= -(-ie)^2 \frac{[\bar{v}_2\gamma^{\mu}u_1][\bar{U}_1'\gamma_{\mu}V_2']}{s} \tag{4.2}$$

at tree level, where $s = (p_1 + p_2)^2$ is the usual Mandelstam variable. Shorthand notation $u_i \equiv u^{s_i}(\mathbf{p}_i)$ is being used. The spinors u, v are taken to correspond to the electron, while U, V are associated with the muon.

Question 5

Proposition 5.1: *In the limit of massless fermions, the spin-averaged squared amplitude for the process* $e^+e^- \to \mu^+\mu^-$ *is*

$$\left\langle|\mathcal{A}|^2\right\rangle = \frac{32\pi^2\alpha^2}{s^2}(s^2 + 2st + 2t^2). \tag{5.1}$$

Proof.—We square the amplitude written down in Question 4 to obtain

$$|\mathcal{A}|^2 = \frac{e^4}{s^2}[\bar{v}_2\gamma^{\mu}u_1][\bar{u}_1\gamma^{\nu}v_2][\bar{U}_1'\gamma_{\mu}V_2'][\bar{V}_2'\gamma_{\nu}U_1']. \tag{5.2}$$

Upon taking the spin average, we can immediately exploit the spin-sum relations to write $\sum u_1\bar{u}_1 = \not{p}_1 + m_e$ and $\sum V_2'\bar{V}_2' = \not{p}_2' - m_{\mu}$. Keeping in mind that our ultimate goal is to compute a scattering cross section to be compared with experiment, let us neglect the masses of the fermions in what follows. This simplification is justified since the experiment, PETRA,[6] collides electrons and positrons with centre-of-mass energy on the order of tens of GeV's; much greater than $m_e = 511\,\text{keV}$ and $m_{\mu} = 106\,\text{MeV}$. Hence,

$$\begin{aligned}\left\langle|\mathcal{A}|^2\right\rangle &= \frac{e^4}{4s^2}\sum_{\text{spins}}[\bar{v}_2\gamma^{\mu}\not{p}_1\gamma^{\nu}v_2][\bar{U}_1'\gamma_{\mu}\not{p}_2'\gamma_{\nu}U_1'] \\ &= \frac{e^4}{4s^2}\operatorname{tr}(\not{p}_2\gamma^{\mu}\not{p}_1\gamma^{\nu})\operatorname{tr}(\not{p}_1'\gamma^{\mu}\not{p}_2'\gamma^{\nu}).\end{aligned} \tag{5.3}$$

The second line follows from using the identity $\bar{\psi}A\psi = \operatorname{tr}(\psi\bar{\psi}A)$, valid for any spinor ψ and 4×4 matrix A, and then using the spin-sum relations again.

To proceed any further requires knowing some traces of Dirac matrices. We dust off our work from Sheet 3 to recall that

$$\operatorname{tr}(\gamma^{\alpha}\gamma^{\mu}\gamma^{\beta}\gamma^{\nu}) = 4(\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\mu\beta} - \eta^{\alpha\beta}\eta^{\mu\nu}). \tag{5.4}$$

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[6] The Positron-Electron Tandem Ring Accelerator at DESY in Hamburg, Germany.

Contracting this with $(p_2)_\alpha$ and $(p_1)_\beta$ yields

$$\text{tr}(\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu) = 4[p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - (p_1 \cdot p_2)\eta^{\mu\nu}] = 4[2p_1^{(\mu}p_2^{\nu)} - (p_1 \cdot p_2)\eta^{\mu\nu}]. \tag{5.5}$$

Doing the same for the other trace, one finds

$$\langle|\mathcal{A}|^2\rangle = \frac{16e^4}{s^2}p_1^{(\mu}p_2^{\nu)}p'_{1,\mu}p'_{2,\nu} = \frac{8e^4}{s^2}[(p_1 \cdot p'_1)(p_2 \cdot p'_2) + (p_1 \cdot p'_2)(p_2 \cdot p'_1)]. \tag{5.6}$$

Our next step is to use the identities

$$s = 2p_1 \cdot p_2, \quad t = -2p_1 \cdot p'_1, \quad u = -2p_1 \cdot p'_2; \tag{5.7}$$

which all hold only in the massless limit. These imply that

$$\langle|\mathcal{A}|^2\rangle = \frac{2e^4}{s^2}(t^2 + u^2) = \frac{2e^4}{s^2}(s^2 + 2st + 2t^2). \tag{5.8}$$

The second expression follows from using the identity $s + t + u = 0$ to eliminate u. Rewriting e in terms of the fine-structure constant $\alpha = e^2/(4\pi)$ yields the desired result. ■

Proposition 5.2: *The spin-averaged cross section for this process is*

$$\sigma = \frac{4\pi\alpha^2}{3s}. \tag{5.9}$$

Proof.—As always, our starting point is the master formula for the differential scattering cross section [3],

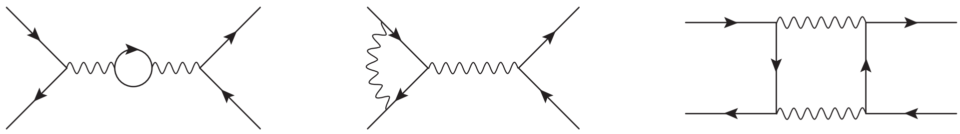
$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{|\mathcal{A}|^2}{16\pi\lambda(s, m_1^2, m_2^2)}, \tag{5.10}$$

where $m_{1,2}$ are the masses of the initial particles, and $\lambda(s, m_1^2, m_2^2) = [s-(m_1+m_2)^2][s-(m_1-m_2)^2]$. Setting $m_1 = m_2 = 0$ and using the result of Proposition 5.1, we obtain

$$\sigma = \int \mathrm{d}t \, \frac{2\pi\alpha^2}{s^4}(s^2 + 2st + 2t^2). \tag{5.11}$$

In Question 8 of Sheet 3, we previously established that the Mandelstam variable $t \in [-(s-4m^2), 0]$ if all four external particles have the same mass m. Setting $m = 0$ and performing the integral over t yields the desired result. ■

It is natural to wonder if this tree-level calculation suffices for comparison with data. In other words, what is the theoretical uncertainty that we accrue by neglecting higher-order (loop) corrections? Several examples of one-loop diagrams are shown below:



Just by counting the number of vertices present, we see that the one-loop diagrams contribute terms proportional to $e^4 \propto \alpha^2$ to the amplitude. Thus, one-loop corrections to σ are proportional to α^4, meaning they are smaller than the tree-level terms by about $\alpha^2 \sim 10^{-4}$. This is well below the threshold of PETRA's experimental uncertainty, so can be neglected.

Appendix: Common Feynman rules

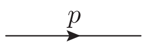
In this course, we encountered three different types of quantum fields: spin-zero scalars, spin-half spinor fields (fermions), and the spin-one Maxwell field (photon). Feynman rules for their propagators and external legs are always the same, so let us collect them here for convenience.

Propagators — A scalar of mass μ is denoted by a dashed line, and each internal propagator comes with a factor

$$= \frac{i}{p^2 - \mu^2 + i\epsilon}.$$

The arrow denotes the direction of flow of momentum, and also serves to distinguish particle from antiparticle if the scalar is complex.

A fermion of mass m is denoted by a solid line, and each internal propagator yields



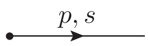
$$= \frac{i(\not{p} + m)}{p^2 - \mu^2 + i\epsilon}.$$

Finally, a wavy line represents a photon, whose propagator is

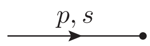
$$= \frac{-i\eta^{\mu\nu}}{p^2 + i\epsilon}$$

when written in Feynman (Lorenz) gauge. This gauge choice is invariably the simplest for calculations.

External legs — For each ingoing or outgoing fermion, we attach the following spinors:

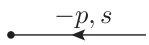


$= u^s(\mathbf{p}),$

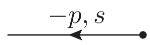


$= \bar{u}^s(\mathbf{p}).$

Similarly, for each ingoing our outgoing antifermion, we write



$= \bar{v}^s(\mathbf{p}),$



$= \bar{v}^s(\mathbf{p}).$

Note that, by convention, we use the arrows on the solid line to distinguish between particle and antiparticle, and also to represent the direction of momentum flow. In all cases above, momentum p is flowing from left to right, hence momentum $-p$ is flowing in the direction of the arrows of the antifermions.

Analogously, each ingoing or outgoing photon is multiplied by a polarization vector,

p, λ  $= \mathcal{E}_\mu^{\lambda *}(\mathbf{p}),$ $p, \lambda = \mathcal{E}_\mu^{\lambda}(\mathbf{p}).$

Scalars, either real or complex, do not have any factors associated with their external legs.

Miscellanea — A few other rules complete the list:

- Conserve momentum at each vertex. This automatically ensures that the total momentum for the ingoing and outgoing particles is conserved.
- To ensure that the spinor indices are contracted correctly, for each solid line begin at the head and systematically work backwards until reaching the tail.
- Occasionally, there are extra minus signs that need to be included due to Fermi statistics. This is best illustrated with an example; see Question 1.
- Divide each diagram by its appropriate symmetry factor.

References

[1] M. E. Peskin, and D. V. Schroeder, *An Introduction To Quantum Field Theory* (Perseus Books Publishing, 1995).

[2] M. Srednicki, *Quantum Field Theory* (Cambridge University Press, Cambridge, England, 2007).

[3] B. C. Allanach, *QFT: Cross Sections and Decay Rates*, (2018) http://www.damtp.cam.ac.uk/user/examples/3P1l.pdf.