The much-neglected second normal stress difference

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This article combines my early interest in how microstructure leads to rheology and my later interest in how rheology leads to flow behaviour.

As there are different definitions in the literature, let me recall the definitions of normal stresses that I shall use. In a simple shear $\mathbf{u} = (\gamma y, 0, 0)$, the stress tensor has 5 non-zero components, the tangential viscous stresses $\sigma_{xy} = \sigma_{yx}$, and the non-dissipative normal stresses σ_{xx}, σ_{yy}, and σ_{zz}. In a Newtonian fluid, the normal stress components are all equal, to minus the pressure, but in a visco-elastic fluid, the components are not equal. The first normal stress difference is $N_1 = \sigma_{xx} - \sigma_{yy}$, which when positive can be thought of as a tension in the streamlines in the x-direction, or alternatively a pressure in the flow-gradient y-direction. I shall take the second normal stress difference as $N_2 = \sigma_{yy} - \sigma_{zz}$, which when negative can be thought of as a tension in the vortex lines in the z-direction, or alternatively a pressure again the flow-gradient y-direction.

When I learnt about rheology fifty years ago, the important visco-elastic fluids were polymer melts in industry and polymer solutions in university laboratories. These have a positive N_1, and an N_2 so small that it was difficult to measure, and hence was nearly always neglected. Thinking of N_1 as a tension in the streamlines explains simply many phenomena:– the Weissenberg rod-climbing effect, the migration of particles to the centre line of pipe flows, the stabilisation of jets in air, the purely-elastic instability of Taylor-Couette flow, an instability in co-extrusion, and more. It is useful to think about the origin of N_1 for polymers, and other microstructures composed of fibres. In a simple shear flow, the fibres are first stretched by the flow and then aligned with the flow. Thus a tension is generated in the streamlines. Further, this mechanism does not generate any N_2.

To have a non-zero N_2, one needs a thick microstructure that can be compressed, e.g. droplets in an emulsion. Simple shear flow first stretches the microstructure in the first and third quadrants ($xy > 0$) and compresses it in the second and fourth quadrants (with $xy < 0$). The vorticity then rotates the microstructure to give stretching in the flow direction, i.e. $N_1 > 0$, and compression in the flow-gradient direction, i.e. $N_2 < 0$.

While thick microstructures unavoidably rotate fully with the vorticity, they do not strain fully with the strain-rate, because of their internal resistance to deformation. This leads to non-affine deformations. If the straining is reduced to an efficiency of $\theta < 1$, then the appropriate time derivative is a mix of $\frac{1}{2}(1+\theta)$ of the upper-convected derivative and $\frac{1}{2}(1-\theta)$ of the lower-convected derivative. This change to non-affine deformation produces a shear-thinning rheology along with $N_2 < 0$.

Another origin of N_2 occurs in non-Brownian suspensions. In a simple shear flow, the particles impact one another in the flow x-direction, producing a pushing $\sigma_{xx} < 0$. When the concentration exceeds 20%, they also impact particles in the layers above and below them, leading to a similar pushing in the flow-gradient y-direction, $\sigma_{yy} < 0$. Force-chains are found at 45° to the flow, i.e. $\sigma_{yy} \approx \sigma_{xx}$, so $N_1 \approx 0$. Instead of passing under and over one another, the particles can pass more easily in the vorticity z-direction, so $\sigma_{zz} \approx 0$, so $N_2 = \sigma_{yy} - \sigma_{zz} < 0$, see Boyer, Pouliquen & Guazzelli (2017).

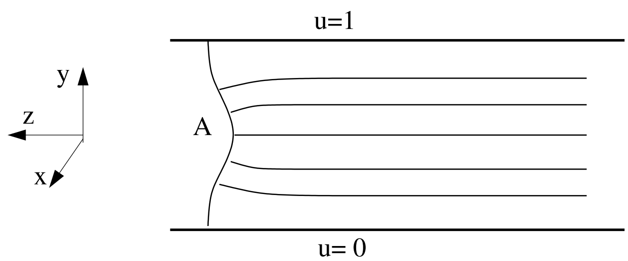
A negative N_2 should be thought of as a tension in the vortex lines. This idea allows one to give simple explanations of several non-Newtonian phenomena:– the bowing of the interface in Tanner's tilted channel, longitudinal vortices in the flow of grains down a chute, negative rod-climbing, an edge instability in rheometers, and lopsided de-wetting on a vertical fibre.

In 1974 Kuo & Tanner studied flow down an inclined open channel with a circular cross-section. I think that it is easier to analyse a shallow cross-section. In the centre of the channel, the flow is faster, and in fact the shear-rate is higher. This higher shear-rate means higher tension in the vortex lines in the centre. This higher tension pulls fluid into the centre. The surface therefore bows up, due only to N_2. I speculate that the same mechanism would explain the longitudinal vortices observed in granular chute flow by Forterre & Pouliquen (2017), who provided an alternative explanation.

The standard Beavers & Joseph (1975) analysis of the Weissenberg rod-climbing finds that the liquid surface will climb the rotating rod if $N_1 + 4N_2 >$

0. For polymers, $N_1 > 0$ and $N_2 \approx 0$, so there is rod-climbing through the squeezing of the hoop-stress from the tension in the streamlines. On the other hand for concentrated non-Brownian suspensions, $N_1 \approx 0$ and $N_2 < 0$ so the free surfaces dips near the rotating rod, through tension in the vertical vortex lines, see Boyer, Pouliquen & Guazzelli (2017).

Sometimes visco-elastic fluids are ejected from a cone-and-plate rheometer, not through inertia, but through a purely elastic instability associated with N_2. This so-called 'edge instability' was first investigated by Tanner in 1993, and recently more thoroughly by Hemmingway & Fielding in 2017. Consider the figure looking at the cross-section of the edge of the rheometer,



with the top plate moving out of the paper in the x-direction and the bottom plate stationary. The horizontal lines between the two plates are contours of constant u velocity in the x-direction. They are also vortex lines. Consider a perturbation of the free surface between the air and the liquid, with a dimple A into the liquid in the middle between the plates. To avoid a shear stress exerted across the free surface, the contours of u must meet the free surface at right angles. This concentrates the contours near the dimple A. This increases the shear-rate near A. This creates a higher tension in the vortex lines at A. This pulls the perturbation A further into the liquid, i.e. there is an instability.

My final example of a flow behaviour resulting from N_2 concerns the draining under gravity of a thin film on a circular vertical fibre. First reported by Boulogne in his 2013 Paris thesis, a visco-elastic liquid can move to one side of the fibre de-wetting the other side. I looked at this with Claire McIlroy in 2015. Consider starting with a film of uniform thickness around the fibre. Now add a perturbation in the thickness which makes one side slightly thicker. The liquid will drain faster on the thicker side. In fact, the shear-rate is higher on the thicker side. So on the thicker side there is a higher tension in the vortex lines, which are circular lines around the fibre. The higher tension acts to pull more liquid round to the thicker side, i.e. there is a de-wetting instability. In the final stages of de-wetting, there is an interesting $t^{-1/4}$ slow drainage.

To conclude, the early days of visco-elastic fluids were dominated by polymeric liquids, for which the second normal stress was negligibly small and hence often appropriately neglected. Today there is more interest and more applications in non-polymeric liquids, such as emulsions and complex suspensions. These have significant second normal stresses. And these second normal stresses lead to interesting and curious flow behaviours. The second normal stress difference should no longer be neglected.