Particles impacting on a granular bed

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Abstract. An asymptotic analysis is made to find the penetration depth and the stopping time for a particle impacting a granular bed. Newton's equation is solved with a drag force with two terms, one term proportional to the square of the velocity and one term linear in the depth. The penetration depth is found to increase with the logarithm of the impact velocity, while the stopping time is found to decrease with the inverse of the square root of the logarithm of the impact velocity.

1. Introduction

When a particle impacts a granular bed, how deep does it penetrate the bed, and how quickly does it stop? Answering these questions may help understand the formation of crater when asteroids and military objects impact the Earth. In the last 10 years, there have been a number of laboratory studies of particles impacting granular beds, along with a few simplified numerical simulations. For spheres of diameter D and density ρ_s falling freely a height H and then impacting a granular bed of density ρ_b, the depth of penetration δ has been found to scale as

$$\frac{\delta}{D} \propto \left(\frac{\rho_s}{\rho_b}\right)^{\alpha} \left(\frac{H}{D}\right)^{\beta},$$

with various values of the indices α and β reported :– (Uehara et. al., 2003), (de Bruyn & Walsh, 2004) and (Ambroso et. al., 2005) found in experiments $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$, (Tsimring & Volfson, 2005) found in two-dimensional simulations $\alpha = \beta = \frac{2}{5}$ which they suggested in three dimensions should be $\alpha = \beta = \frac{1}{3}$, and (Goldman & Umbanhowar, 2008) found in experiments $\alpha = \beta = \frac{1}{2}$. The stopping time exhibits a curious behaviour of faster impacting particles stopping in a shorter time, tending to non-zero plateau value at high impacting velocities, $H \gtrsim D\rho_s/\rho_b$. (Ciammara et. al., 2004) saw only the plateau value in their experiments and two-dimensional simulations, whereas (Goldman & Umbanhowar, 2008) suggest from experiments a plateau value scaling with $(\rho_s/\rho_b)^{1/4}(D/g)^{1/2}$ and (Seguin et. al., 2009) suggest from simulations $1.7(\rho_s/\rho_b)^{1/2}(D/g)^{1/2}$. The precise scalings of the penetration depth and the stopping time thus remain unclear.

On the other hand, a consensus has emerged on the form of the resistive force that a granular bed exerts on a moving particle. The force is the sum of two parts, and a fluid-like inertial part proportional to the square of the instantaneous velocity and a dry-solid friction part proportional to the instantaneous depth z in the bed, i.e.

$$F = -C_D \rho_b D^2 v^2 - \mu D^2 \rho_b g z, \tag{1}$$

where g is the acceleration due to gravity and C_D and μ are dimensionless constants. The two parts resist the motion and so would change sign if the velocity became negative. Moreover for a stationary particle, the dry-solid friction term may be less that $\mu D^2 \rho_b g z$. The inertial part, first suggested from two-dimensional simulations by (Tsimring & Volfson, 2005), is the force required to accelerate from rest to the velocity v a mass $\rho_b D^2 v$ per unit time. The quadratic variation with velocity was confirmed in experiment by (Katsuragi & Durian, 2007) with a single granular bed and a single sphere, who also found that the dependence on depth of the dry-solid part was best approximated by a linear variation in depth, corresponding to a Coulombic coefficient of friction multiplied by the normal pressure force which itself is proportional to depth. They gave values for the dimensionless coefficients in the force law (1) as $C_D = 0.8$ and $\mu = 9$. (Goldman & Umbanhowar, 2008) found in experiments with four different granular beds and 15 different spheres of various densities and radii that at high velocities and shallow-impacts that the force varied quadratically with the velocity, quadratically with the diameter of the sphere, and was independent of the mass of the sphere. Deeper into the bed and at lower velocities, there would have been a significant contribution from the dry-solid friction, which seems not to have been examined separately. In two-dimensional simulations of particles with zero tangential friction, (Seguin et. al., 2009) demonstrated that the inertial part was proportional to the square of the velocity, the density of the bed and the diameter of the particle, and that the dry-solid friction part was proportional to the density of the bed, the diameter of the particle, and linear in the depth in the bed. The linear variation with the diameter in two dimensions becomes the square of the diameter in three dimensions. Using a single bed of very light expanded polystyrene particles and a single hollow sphere whose mass could be varied to 18 different values, (Pacheco-Vázquez et. al., 2011) found that the trajectories of the sphere were consistent with the force law (1) while the sphere penetrated the bed no further than one diameter of the container. Deeper into the bed, the dry-solid friction part of the force tended to a constant, due to the Janssen effect making the pressure tend to a constant deep into the bed. (Pacheco-Vázquez

et. al., 2011) gave values for the dimensionless coefficients $C_D = 2.4$ and $\mu = 12$.

Not withstanding the difference in the values of the dimensionless coefficients, force law (1) seems well established. In each of the studies, it predicts well the trajectories of the particles, and so predicts well the penetration depth and the stopping time. This opens the possibility of settling what are the correct scaling laws for the penetration depth and stopping times by an asymptotic analysis of the equation of motion using the the established force law. That possibility is the subject of this article.

2. Governing equations

The motion of the impacting sphere is governed by Newton's law with the weight of the sphere and the resistive force (1). Let z be the distance downwards from the free surface of the bed. The mass of the sphere is $\rho_s \frac{\pi}{6} D^3$. It is convenient to non-dimensionalise the problem so that the coefficients of the two friction terms are unity. To make the coefficient of the inertial term unity, length is non-dimensionalised by

$$L = \frac{\pi}{6C_D} \frac{\rho_s}{\rho_b} D.$$

To make the coefficient of the dry-solid friction term unity, time is non-dimensionalised by

$$T = \sqrt{\frac{\pi}{6\mu} \frac{\rho_s}{\rho_b} \frac{D}{g}}.$$

The governing equation then becomes

$$\ddot{z} = k - z - \dot{z}^2, \tag{2}$$

where $k = C_D/\mu$ in the term which represents the weight of the particle, having made the two coefficients in the friction law equal to unity. The experiments of (Katsuragi & Durian, 2007) give $k = 0.09$, while the more recent experiments by (Pacheco-Vázquez et. al., 2011) give $k = 0.2$. For the purpose of plotting results in this article, the value $k = 0.2$ will be used.

The initial conditions are that the sphere starts at the surface with a velocity from falling freely through the height H, which with the above non-dimensionalisation become

$$z(0) = 0 \quad \text{and} \quad \dot{z}(0) = V_0 = \sqrt{\frac{12\mu}{\pi C_D^2} \frac{\rho_b}{\rho_s} \frac{H}{D}}. \tag{3}$$

We shall be interested in the asymptotic behaviour for large impact velocities $V_0 \gg 1$. The largest value in experiments is around $V_0 = 10$, corresponding to dropping a 2.5 cm sphere 1 m and the density of the sphere being twice that of the bed.

3. Direct integration

It is possible to integrate the governing equation twice to obtain expressions for the penetration distance and stopping time. The expressions can then be evaluated asymptotically in the limit of large impact velocities. The nonlinear governing equation (2) can be simplified by making a Riccati inspired transformation. Introducing

$$z = \ln x, \qquad \text{so} \quad \dot{z} = \frac{\dot{x}}{x} \quad \text{and} \quad \ddot{z} = \frac{\ddot{x}}{x} - \frac{\dot{x}^2}{x^2},$$

the governing equation becomes

$$\ddot{x} = x(k - \ln x).$$

This has a first integral

$$\dot{x}^2 = V_0^2 + (k + \tfrac{1}{2})(x^2 - 1) - x^2 \ln x, \tag{4}$$

using initial conditions $x = 1$ and $\dot{x} = V_0$ from (3).

The velocity vanishes at x_∞ given by

$$x_\infty^2(\ln x_\infty - k - \tfrac{1}{2}) = V_0^2 - k - \tfrac{1}{2}.$$

For $V_0 \gg 1$, one can solve this iteratively for

$$x_\infty \sim \frac{V_0}{\sqrt{\ln V_0}} \left[1 + \frac{1}{2\ln V_0}\left(\ln\sqrt{\ln V_0} + k + \tfrac{1}{2}\right)\right]. \tag{5}$$

Thus the penetration depth is

$$z_\infty \sim \ln V_0 - \ln\sqrt{\ln V_0} + \frac{1}{2\ln V_0}\left(\ln\sqrt{\ln V_0} + k + \tfrac{1}{2}\right). \tag{6}$$

We shall treat $\ln\sqrt{\ln V_0}$, which occurs in this expression and in many subsequent expressions, as an $O(1)$ quantity.

Figure 1 plots the penetration depth z_∞ as a function of impact velocity V_0, comparing the asymptotic approximation (6) with results of numerical solution of equation (2). Typical of expansions involving logarithms, the first approximation $z_\infty \sim \ln V_0$ is vaguely nearby when the small expansion parameter $1/\ln V_0 = 0.9$ ($V_0 = 3$) is quite large,

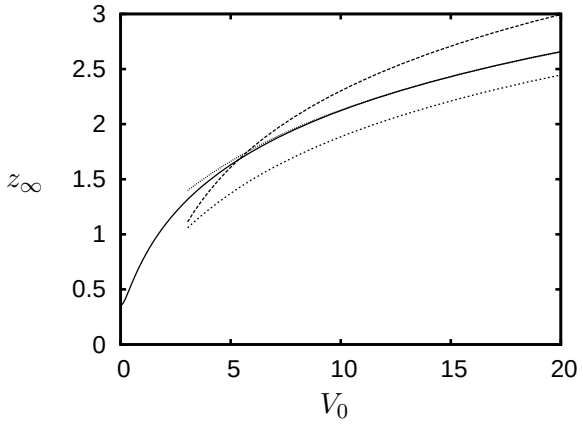

Figure 1. Penetration depth z_∞ as a function of impact velocity V_0. The continuous curve is the result of a numerical integration of equation (2). The long dashed curve is the first term of the asymptotic result (6), while the short dashed curve is the first two terms and the dotted curve is all three terms.

but has a relative error around 15% at least to $V_0 = 1000$. Adding the second term, $-\ln\sqrt{\ln V_0}$, leaves the relative error at 8% for $V_0 = 20$ but the error does now decrease if only very slowly, becoming 4% by $V_0 = 100$ and 2% by $V_0 = 1000$. Adding the final term dramatically improves the accuracy to around 0.2% through this range.

The stopping time t_∞ requires an integration of the first integral (4)

$$t_\infty = \int_1^{x_\infty} \left[V_0^2 - k - \tfrac{1}{2} - x^2(\ln x - k - \tfrac{1}{2})\right]^{-1/2}\, dx.$$

To evaluate this at large V_0, it is useful to substitute $x = x_\infty \xi$ to give

$$t_\infty = \frac{1}{\sqrt{\ln x_\infty - k - \frac{1}{2}}} \int_{1/x_\infty}^1 \left[1 - \xi^2\left(1 + \frac{\ln \xi}{\ln x_\infty - k - \frac{1}{2}}\right)\right]^{-1/2} d\xi.$$

When $x_\infty \gg 1$ this gives

$$t_\infty \sim \frac{\pi/2}{\sqrt{\ln x_\infty - k - \frac{1}{2}}} \left(1 + \frac{\frac{1}{2}(1 - \ln 2)}{\ln x_\infty - k - \frac{1}{2}}\right).$$

By using our asymptotic expression for x_∞ (5), the stopping time in terms of V_0 is

$$t_\infty \sim \frac{\pi}{2\sqrt{\ln V_0}} \left[1 + \frac{1}{2\ln V_0}(\ln\sqrt{\ln V_0} + k - \tfrac{1}{2} + \ln 2)\right]. \qquad (7)$$

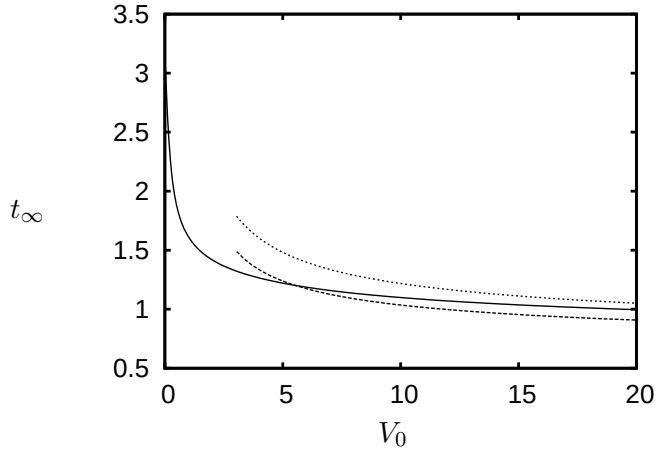

Figure 2. Stopping time t_∞ as a function of impact velocity V_0. The continuous curve is the result of a numerical integration of equation (2). The dashed curve is the first term of the asymptotic result (7), while the dotted curve includes the correction.

Figure 2 plots the stopping time t_∞ as a function of impact velocity V_0, comparing the asymptotic approximation (7) with results of numerical solution of equation (2). As with the penetration depth, the leading approximation $t_\infty \sim \pi/2\sqrt{\ln V_0}$ is nearby throughout the range, but the error of around 0.1 does not decrease significantly. On the other hand, the second approximation gradually improves with an error of 0.06 at $V_0 = 20$ decreasing to 0.008 at $V_0 = 100$.

The asymptotic results for the penetration depth and the stopping time involve logarithms. The presence of logarithms explains the difficulty experienced in previous studies when trying to fit power-law scalings. We shall return to this issue in the discussion section.

The above direct integration yields expressions for the penetration depth and the stopping time, but gives no insight into the form of the expressions, and in particular fails to explain why faster particles stop in a shorter time. To generate an understanding, we solve the problem with matched asymptotic expansions. There is first a fast phase when the velocity is $O(V_0)$ and the inertial part of the resistive force is dominant. The particle comes to rest in a second phase during which the two parts of the resistive force are comparable and the position changes little.

4. Fast initial phase

4.1. Scaling

During the first phase, the velocity is large, $\dot{z} = O(V_0)$, while the displacement is modest, $z = O(1)$, so the time scale must be short, $O(1/V_0)$. Hence we introduce a fast time scale

$$\tau = V_0 t.$$

The governing problem then becomes

$$z_{\tau\tau} = -z_\tau^2 + \frac{1}{V_0^2}(k - z) \quad \text{with} \quad z_\tau(0) = 1.$$

In this initial phase the velocity is high, so that the inertial drag dominates.

The above form of the equation suggests an expansion

$$z(t, V_0) \sim \zeta_1(\tau) + \frac{1}{V_0^2}\zeta_2(\tau).$$

4.2. First approximation

The leading order term is governed by

$$\zeta_{1\tau\tau} = -\zeta_{1\tau}^2, \quad \text{with} \quad \zeta_1(0) = 0, \quad \zeta_{1\tau}(0) = 1.$$

The solution is

$$\zeta_{1\tau} = \frac{1}{1+\tau} \quad \text{and} \quad \zeta_1 = \ln(1+\tau).$$

4.3. Correction

The correction is governed by

$$\zeta_{2\tau\tau} + 2\zeta_{1\tau}\zeta_{2\tau} = k - \zeta_1 \quad \text{with} \quad \zeta_2(0) = 0, \quad \zeta_{2\tau}(0) = 0.$$

The solution is

$$\zeta_2 = \tfrac{1}{6}(1+\tau)^2\left(-\ln(1+\tau) + k + \tfrac{5}{6}\right) + \tfrac{1}{3}(k + \tfrac{1}{3})/(1+\tau) - \tfrac{1}{2}k - \tfrac{1}{4}.$$

4.4. Asymptoticity broken

The initial fast phase comes to an end when the increasing dry-solid friction term $-z$ in the governing equation becomes comparable with the decreasing inertial friction term $-\dot{z}^2$, i.e. when

$$\ln(1+\tau) = O\left(\frac{V_0^2}{\tau^2}\right),$$

i.e. at the large

$$\tau = O\left(\frac{V_0}{\sqrt{\ln V_0}}\right).$$

This large value of the initial fast time scale is the small time $t = O(1/\sqrt{\ln V_0})$. At this time the correction to the velocity $\zeta_{2\tau}/V_0^2$ becomes comparable to the first approximation to the velocity $\zeta_{1\tau}$ and

$$\dot{z} = O\left(\sqrt{\ln V_0}\right).$$

On the other hand the displacement is still dominated by the first approximation ζ_1

$$z = O\left(\ln V_0\right).$$

5. Final stopping phase

5.1. Scalings

The break down of the initial fast phase sets the scalings of the final phase. We introduce a slower fast time

$$T = \sqrt{\ln V_0}\, t$$

and let Z be the $O(1)$ change in the large displacement established in the initial fast phase

$$z(t) = \ln V_0 + Z(T)$$

The governing equation then becomes

$$Z_{TT} = -1 - Z_T^2 + \frac{1}{\ln V_0}\left(k - Z\right).$$

5.2. Preparing to match backwards

Expressing the end of the initial phase at large τ in terms of the new slower fast time scale T

$$z \sim \ln V_0 - \ln\sqrt{\ln V_0} + \ln T - \tfrac{1}{6}T^2 + \frac{1}{6\ln V_0}(\ln\sqrt{\ln V_0} - \ln T + k + \tfrac{5}{6}) \quad \text{as} \quad T \searrow 0.$$

We are treating the $\ln\sqrt{\ln V_0}$ as an $O(1)$ quantity. The governing equation and the matching both then suggest an expansion for the final stopping phase as

$$Z \sim Z_1 + \frac{1}{\ln V_0} Z_2.$$

5.3. First approximation

The first approximation is governed by

$$Z_{1TT} = -1 - Z_{1T}^2,$$

with

$$Z_1 \sim -\ln\sqrt{\ln V_0} + \ln T - \tfrac{1}{6}T^2 \quad \text{as} \quad T \searrow 0.$$

The solution is

$$Z_{1T} = \cot T \quad \text{and} \quad Z_1 = -\ln\sqrt{\ln V_0} + \ln(\sin T),$$

with constants of integration set by the matching.

5.4. Correction

The correction is governed by

$$Z_{2TT} + 2Z_{1T}Z_{2T} = k - Z_1$$

with

$$Z_2 \sim \tfrac{1}{6}T^2(\ln\sqrt{\ln V_0} - \ln T + k + \tfrac{5}{6}) \quad \text{as} \quad T \searrow 0$$

The solution for Z_{2T} is

$$Z_{2T} = \csc^2 T \int_0^T \sin^2 s \left(\ln\sqrt{\ln V_0} + k - \ln(\sin s)\right)\, ds$$

with a constant of integration set by matching. Integrating,

$$Z_2 = \int_0^T (\cot s - \cot T)\sin^2 s \left(\ln\sqrt{\ln V_0} + k - \ln(\sin s)\right)\, ds$$

again with a constant of integration set by matching.

5.5. Stopping time

The velocity will vanish at the slow fast time

$$t_\infty = \frac{1}{\sqrt{\ln V_0}} T_\infty \quad \text{where} \quad Z_T(T_\infty) = 0.$$

Now for a first approximation for T_∞, we note

$$Z_{1T} = 0 \quad \text{at} \quad T = \frac{\pi}{2}.$$

Hence for a better approximation, we pose

$$T_\infty \sim \frac{\pi}{2} + \frac{1}{\ln V_0} \Delta T.$$

Then

$$Z_T(T_\infty) \sim Z_{1T}(\tfrac{\pi}{2}) + \frac{1}{\ln V_0} \left(Z_{1TT}(\tfrac{\pi}{2}) \Delta T + Z_{2T}(\tfrac{\pi}{2}) \right).$$

Now $Z_{1TT} = -1 - Z_{1T}^2$ and $Z_{1T}(\frac{\pi}{2}) = 0$, so

$$\Delta T = Z_{2T}(\tfrac{\pi}{2}) = \tfrac{\pi}{4} \left(\ln \sqrt{\ln V_0} + k - \tfrac{1}{2} + \ln 2 \right).$$

Hence

$$t_\infty \sim \frac{1}{\sqrt{\ln V_0}} \frac{\pi}{2} + \frac{1}{(\ln V_0)^{3/2}} \frac{\pi}{4} \left(\ln \sqrt{\ln V_0} + k - \tfrac{1}{2} + \ln 2 \right).$$

This agrees with expression (7) obtained in section 3.

5.6. Penetration depth

Evaluating the depth at the stopping time

$$z(t_\infty) \sim \ln V_0 + Z_1(\tfrac{\pi}{2}) + \frac{1}{\ln V_0} \left(Z_{1T}(\tfrac{\pi}{2}) \Delta T + Z_2(\tfrac{\pi}{2}) \right).$$

Now $Z_1(\frac{\pi}{2}) = -\ln \sqrt{\ln V_0}$ and $Z_{1T}(\frac{\pi}{2}) = 0$ and $Z_2(\frac{\pi}{2})$ can be evaluated simply, giving

$$z(t_\infty) \sim \ln V_0 - \ln \sqrt{\ln V_0} + \frac{1}{\ln V_0} \left(\tfrac{1}{2} (\ln \sqrt{\ln V_0} + k + \tfrac{1}{2}) \right).$$

This agrees with expression (6) obtained in section 3.

6. Discussion

The asymptotic analysis of §§4 & 5 reveals how the particle stops. While the velocity is large, the inertial part of the resistive force dominates, so

$$\ddot{z} \sim -\dot{z}^2,$$

with solution

$$\dot{z} \sim \frac{V_0}{1+V_o t} \quad \text{and} \quad z \sim \ln(1+V_0 t).$$

The two terms in the resistive force eventually become comparable, when

$$\left(\frac{V_0}{1+V_o t}\right)^2 \sim \ln(1+V_0 t),$$

i.e. at a time

$$t \sim \frac{1}{\sqrt{\ln V_0}}, \qquad \text{when} \quad \dot{z} = O(\sqrt{\ln V_0}) \quad \text{and} \quad z \sim \ln V_0.$$

Thereafter the inertial term drops to zero as the particle stops while the dry-solid friction term stays asymptotically constant, producing a deceleration $-\ln V_0$. This deceleration will finally stop the particle moving at a speed $O(\sqrt{\ln V_0})$ in a time $\pi/2\sqrt{\ln V_0}$, with the particle advancing only a $O(1)$ distance during this time. Hence the penetration depth increases logarithmically with impact velocity V_0, while the stopping time decreases as $\pi/2\sqrt{\ln V_0}$.

This behaviour is special to the combination of the two terms in the resistive force: with just one of the two terms the outcome is different. If there were only the quadratic inertial term, then the particle would never stop while the penetration would increase indefinitely in time. If there were only the dry-solid friction term, the motion would be a simple harmonic motion. Thus at low impact speeds, the penetration depth would be $2k$ while the stopping time is π, and at high impact speeds the penetration depth increases linearly with V_0 and the stopping time tends down to the plateau value $\pi/2$. It is only with the combination of the two terms does the stopping time continue to decrease as the impact velocity increases, albeit very slowing as the inverse of the square root of the logarithm.

Translating the results of the asymptotic analysis into the original dimensional variables of the problem, the penetration depth is predicted at leading order to be

$$\frac{\delta}{D} \sim c_1 \frac{\rho_s}{\rho_b} \ln\left(\frac{\rho_b}{\rho_s}\frac{H}{D}\right),$$

while the stopping time is predicted to be

$$t_\infty \sqrt{\frac{g}{D}} \sim c_2 \left(\frac{\rho_s}{\rho_b}\right)^{1/2} \left(\ln \frac{\rho_b}{\rho_s}\frac{H}{D}\right)^{-1/2},$$

with two constants c_1 and c_2.

The existence of the logarithms explains the variability in the previous results for the exponents when fitting power-laws to the experimental observations. Indeed if a power-law is fitted to the the numerical results for the penetration depth in figure 1, then one finds different exponents if different ranges are used. Fitting over the range $1 < V_0 < 2$, one finds $\alpha = 0.75$ and $\beta = 0.25$. On the other hand, if one fits over the range $5 < V_0 < 20$ one finds $\alpha = 0.83$ and $\beta = 0.17$. The same ranges fitted to the stopping times in figure 2 produces very similar exponents, so $t_\infty \propto (\rho_s/\rho_g)^{0.42}(D/H)^{0.08}$, close to the scaling suggested by (Seguin et. al., 2009). The power-laws for the penetration depth are different to those found by fitting experimental observations. It may be that the force law needs refinement.

Acknowledgements

Milt was a friend who inspired us to exploit any small parameter and learn.

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