J. Fluid Mech. (1980), *vol.* 101, *part* 2, *pp.* 403–421
Printed in Great Britain

Heat transfer to a slowly moving fluid from a dilute fixed bed of heated spheres

By A. ACRIVOS
Department of Chemical Engineering, Stanford University, California 94305

E. J. HINCH AND D. J. JEFFREY
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW

(Received 26 March 1980)

Using the method of averaged equations, we examine the difference in temperature between the bulk and fixed heated spherical particles under conditions in which ϕ the volume fraction of the particles and ϵ the Peclet number of the flow past the particles are both small. If $\phi \ll \epsilon^2$ the particles are effectively isolated, and so their excess temperature has an $O(\epsilon)$ correction to the pure conduction estimate. On the other hand if $\phi \gg \epsilon^2$, the bulk heating is of sufficient magnitude to produce a significant temperature gradient throughout the fixed bed. This temperature gradient leads to an $O(\phi^{\frac{1}{2}})$ correction to the pure conduction estimate of the excess temperature of the particles, and the correction depends on the details of the flow even though its magnitude is independent of ϵ. A study of the leading-order terms when ϕ and ϵ^2 are of the same magnitude finds that the two small effects are not simply additive.

1. Introduction

The rate of heat or mass transfer from particles in a fixed bed to the surrounding fluid plays an important role, and is often the limiting factor, in a variety of physical operations such as ion exchange, chromatography and the burning of coal. Considerable effort has therefore been devoted to the problem of determining the transfer rates and many empirical correlations – based entirely on experimental data – have been proposed which cover a variety of conditions (for past references, see Sherwood, Pigford & Wilke 1974 and Gunn 1978). In contrast, the theoretical studies of the problem have been few, and have been based exclusively on 'cell' models (Rowe 1963; Pfeffer & Happel 1964; Pfeffer 1964) whose reliability is uncertain.

In the past the development of more rigorous theories was hampered by a dearth of appropriate mathematical tools for dealing with such two-phase systems in which one phase is randomly dispersed. Recently, however, several techniques have appeared which permit a more rigorous approach and which have already yielded some significant results, e.g. when the particles are spheres, expressions for the effective conductivity to $O(\phi^2)$ where ϕ is the volume fraction of the spheres, the effective viscosity to $O(\phi^2)$, the sedimentation velocity to $O(\phi)$ and the force on a particle in a fixed bed to $O(\phi \ln \phi)$ (see the review of Batchelor 1974 for further details). In this paper we shall apply one of these recent methods, specifically the method of averaged equations

0022-1120/80/4655-2530 $02.00

developed by Hinch (1977), to obtain a rigorous theoretical solution, admittedly under restricted conditions, to the problem of heat transfer in a fixed bed.

We shall study the temperature distribution in a bed of uniformly heated fixed spheres with fluid flowing through the bed, and we shall find how much hotter the particles are than the bulk. We shall study the simplest possible case in which the Reynolds number and the Peclet number ϵ of the flow past the particles are small, as is the volume fraction of the particles ϕ. When ϵ is small and ϕ is suitably very small, Acrivos & Taylor's (1962) analysis of a slow flow past an isolated particle should be appropriate. We shall see in §2 that it does apply to the case $\phi \ll \epsilon^2 \ll 1$.

In the other limiting case of exceedingly slow flows, $\epsilon^2 \ll \phi \ll 1$, one might expect that conduction dominates and that to $O(\phi)$ only the interaction between pairs of heated particles need be considered, so that only some average of the conduction solution for two heated spheres, as obtained by Aminzadeh *et al.* (1974), would have to be computed. We shall see in §2, however, that when $\phi \gg \epsilon^2$ the bulk heating is of sufficient magnitude to lead to a significant temperature gradient throughout the fixed bed. In §3 we shall find that, as a result of this temperature gradient, the details of the flow become coupled into the first correction to the isolated particle estimate of the excess temperature of the particles, and that this correction is $O(\phi^{\frac{1}{2}})$ and not the anticipated $O(\phi)$. Both the $O(\phi^{\frac{1}{2}})$ and the following $O(\phi)$ corrections are evaluated in §3 for the case when the thermal conductivities of the particles and the fluid are equal. The slightly more complicated case of unequal conductivities is examined in §4.

We complete our study in §5 with an investigation of the case when ϕ and ϵ^2 are of a comparable small magnitude, and shall show that the small $O(\epsilon)$ correction for $\phi \ll \epsilon^2 \ll 1$ and the $O(\phi^{\frac{1}{2}})$ correction for $\epsilon^2 \ll \phi \ll 1$ are not simply additive.

2. Governing equations

As stated in the introduction, we consider a fixed bed of spherical particles having a radius a and a volume fraction ϕ which is assumed small. The Reynolds number of the flow through the bed is likewise assumed to be small and hence the velocity $\mathbf{u}$ and the pressure p satisfy the Stokes equations

$$\nabla . \mathbf{u} = 0, \quad -\nabla p + \mu \nabla^2 \mathbf{u} = 0, \tag{2.1}$$

where μ is the fluid viscosity. The average velocity, which is equal to the volume flux per unit cross-sectional area of the bed, will be denoted by $\mathbf{U}_0$. Note that to avoid introducing unnecessary complications into many expressions, we use the volume flux per unit area of the complete bed and not the volume flux per unit area of the fluid part of the bed.

The average of the force exerted on the fixed particles has been calculated by Brinkman (1947), Childress (1972) and Howells (1974), as

$$6\pi\mu a\, \mathbf{U}_0[1 + (3/2^{\frac{1}{2}})\,\phi^{\frac{1}{2}} + \tfrac{135}{64}\phi \ln \phi + O(\phi)]. \tag{2.2}$$

The $O(\phi^{\frac{1}{2}})$ term derives from the fact that the average of the velocity disturbance due to a particle in a fixed bed does not satisfy the Stokes equations (2.1) but rather, as we shall see in §3.3, satisfies approximately the so-called Brinkman equations,

$$\nabla . \mathbf{u} = 0, \quad -\nabla p + \mu \nabla^2 \mathbf{u} - \tfrac{9}{2}\mu\phi a^{-2}\mathbf{u} = 0. \tag{2.3}$$

The additional body-force term, $-\frac{9}{2}\mu\phi a^{-2}\mathbf{u}$, represents the force, approximately equal to $-6\pi\mu a\mathbf{u}$, exerted by the other particles which are present in the fixed bed with a number density $3\phi/4\pi a^3$. The solution of the Brinkman equations (2.3) for uniform flow past a fixed sphere is

$$\mathbf{u} = \mathbf{U}_0\left\{1-\frac{3a}{2\kappa^2r^3}[(1+\kappa r+\kappa^2r^2)\,e^{\kappa(a-r)}-(1+\kappa a+\tfrac{1}{3}\kappa^2a^2)]\right\}$$
$$+\tfrac{3}{2}\mathbf{x}(\mathbf{U}_0.\mathbf{x})\,\frac{9}{\kappa^2r^5}[(3+3\kappa r+\kappa^2r^2)\,e^{\kappa(a-r)}-(3+3\kappa a+\kappa^2a^2)], \qquad (2.4)$$

in which $\kappa^2 = \frac{9}{2}\phi a^{-2}$. Thus, as was shown by Brinkman (1947), Childress (1972) and Howells (1974), the other particles in a fixed bed give rise to a screening effect, so that, at large distances from one particle, the velocity disturbance falls off more rapidly than in an infinite fluid, specifically like $U_0 a\kappa^{-2}r^{-3}$ instead of $U_0 ar^{-1}$. This screening is significant at large distances $r \gtrsim \kappa^{-1}$, i.e. $r \gtrsim a\phi^{-\frac{1}{2}}$.

The temperature field in the fixed bed of heated particles is governed by the steady balance between a heat flux $\mathbf{F}$ and heat sources Q

$$\nabla.\mathbf{F} = Q.$$

Inside the particles we take $Q = Q_0$ a constant and in the fluid $Q = 0$. Inside the particles the heat transport is purely conductive with a conductivity k_s, i.e.

$$\mathbf{F} = -k_s\nabla T \quad \text{inside the particles,}$$

while in the fluid, heat is conducted with a conductivity k_f and also is advected by the fluid with a heat capacity ρc_p per unit volume, so that

$$\mathbf{F} = \rho c_p\mathbf{u}T-k_f\nabla T \quad \text{in the fluid.}$$

By extending the definition of $\mathbf{u}$ to be zero inside the particles and by defining k to be the local value of the conductivity (i.e. $k = k_f$ at a point in the fluid and $k = k_s$ at a point in the particle), we may combine the above equations to obtain a single equation valid everywhere for the temperature field

$$\nabla.(\rho c_p\mathbf{u}T) = \nabla.(k\nabla T)+Q. \qquad (2.5)$$

We shall find the average of the temperature field in the fixed bed using averages over the ensemble of realizations of the fixed bed, each realization having different positions of the particles, but the same statistics of their relative positions. Such an ensemble average will be denoted by $\langle T\rangle_0(\mathbf{x})$ and will be called the bulk average. We shall also take conditional averages over those members of the ensemble which have one particle with its centre at a fixed position $\mathbf{x}_1$. This conditional average will be denoted by $\langle T\rangle_1(\mathbf{x}|\mathbf{x}_1)$. Similarly, we shall take conditional averages with two particles fixed at $\mathbf{x}_1$ and $\mathbf{x}_2$, denoting this by $\langle T\rangle_2(\mathbf{x}|\mathbf{x}_1,\mathbf{x}_2)$. Where the arguments of the averaged fields are clear we shall sometimes leave them out. The information about the relative position of the particles which we need in our calculation is the probability distribution of the separation between pairs of particles. We take the simplest case of a uniform probability $3\phi/4\pi a^3$ for separations exceeding $2a$ and zero for separations less than $2a$.

Our problem is thus to solve (2.1) for the velocity field $\mathbf{u}$, substitute this into the temperature equation (2.5) and thence find in average how much hotter the particles are than the bulk, i.e. we wish to evaluate

$$\Delta T(\mathbf{x}) = \frac{3}{4\pi a^3}\int_{|\mathbf{x}_1-\mathbf{x}|<a}[\langle T\rangle_1(\mathbf{x}|\mathbf{x}_1)-\langle T\rangle_0(\mathbf{x})]\,dV_1, \qquad (2.6)$$

where, as we shall see below, the temperature difference ΔT will actually be independent of position in our case. We shall proceed by the method of averaged equations (see, for example, Hinch 1977), i.e. we shall derive an equation governing $\langle T\rangle_1$ and solve it, as opposed to finding first the full temperature field T and then taking an average of that.

The desired quantity ΔT depends on three dimensionless numbers: the volume fraction of particles ϕ, the Peclet number $\epsilon = \rho c_p U_0 a/k_f$ and the ratio of the conductivities $\alpha = k_s/k_f$. We shall assume ϕ and ϵ to be small and allow α to take on arbitrary values.

For an isolated heated sphere at the origin surrounded by stationary fluid, the temperature disturbance created by the particle is

$$T = \begin{cases} \dfrac{Q_0}{k_f}\left(\dfrac{a^2-r^2}{6\alpha}+\dfrac{a^2}{3}\right), & r \leqslant a, \\ \dfrac{Q_0}{k_f}\dfrac{a^3}{3r}, & r \geqslant a, \end{cases}$$

so that

$$\Delta T = \frac{Q_0 a^2}{k_f}\frac{5\alpha+1}{15\alpha} \quad \text{when} \quad \phi = \epsilon = 0.$$

The effect of a slow flow past an isolated particle was studied by Acrivos & Taylor (1962), who found it necessary to consider the interaction of conduction on a length scale of a and advection on a length scale of $k_f/\rho c_p U_0$. For the case of an isolated ($\phi = 0$) and isothermal particle ($\alpha = \infty$) they found that

$$\Delta T = \frac{Q_0 a^2}{3k_f}[1-\tfrac{1}{2}\epsilon-\tfrac{1}{2}\epsilon^2\ln\epsilon-0{\cdot}165\epsilon^2+\tfrac{1}{4}\epsilon^3\ln\epsilon+\ldots], \tag{2.7}$$

i.e. as one might expect, the flow past the particle reduces its temperature. When the particles have a finite conductivity the leading-order term changes, with $(5\alpha+1)/5\alpha$ replacing the initial 1 of the bracket in expression (2.7), but the $O(\epsilon)$ and $O(\epsilon^2\ln\epsilon)$ terms remain unchanged.

There is, however, a fundamental difference between an isolated particle and a fixed bed of particles at a small but non-zero volume fraction, because in a fixed bed there is a non-zero bulk heat source $Q_0\phi$ which renders the bulk temperature field non-uniform. Consequentially, if the fluid does not move ($\epsilon = 0$), this bulk heating forces the bulk temperature to grow at least quadratically in space, and in this case the boundaries of the bed play a dominant role in determining the temperature distribution. On the other hand, if the fluid moves at a very slow velocity U_0 while being heated at a rate $Q_0\phi$, a temperature gradient $Q_0\phi/\rho c_p U_0$ will be established in the direction of $\mathbf{U}_0$. Over the advection length scale $k_f/\rho c_p U_0$ this temperature gradient leads to a temperature drop $Q_0\phi k_f/(\rho c_p U_0)^2$, which is negligible compared with the conduction temperature drop of $Q_0 a^2/k_f$ only if $\phi \ll \epsilon^2$. Hence Acrivos & Taylor's results for an isolated particle can be used in a fixed bed if $\phi \ll \epsilon^2$. In the other limit $\epsilon^2 \ll \phi$, which we shall explore in §§ 3 and 4 of this paper, the bulk heating and the induced bulk temperature gradient cannot be ignored. For simplicity we deal first in § 3 with the case in which the fluid and the particles have equal thermal conductivities ($\alpha = 1$), and then in § 4, add the modifications for unequal conductivities.

An alternative way of seeing that there might be a change in behaviour when ϵ^2

becomes comparable with ϕ is to note that the advection length scale in Acrivos & Taylor's analysis is $a\epsilon^{-1}$ and that the hydrodynamic screening length scale in a fixed bed is $a\phi^{-\frac{1}{2}}$. We explore the transition between the two limiting regions, i.e. we study the case of ϕ comparable to ϵ^2, at the end of this paper in §5.

Before starting our calculation we non-dimensionalize the problem by scaling distances with a, velocities with U_0, forces with $\mu a U_0$, stresses with $\mu U_0/a$, heat sources with Q_0, temperatures with $Q_0 a^2/k_f$ and conductivities with k_f. We also choose the x axis in the direction of the mean flow $\mathbf{U}_0$.

3. Slow flow and equal conductivities: $\epsilon^2 \ll \phi$ and $\alpha = 1$

3.1. *The bulk temperature gradient*

In §2 we saw how the dimensional bulk heat source $Q_0\phi$ induces a dimensional bulk temperature gradient $Q_0\phi/\rho c_p U_0$. This suggests an asymptotic expansion for the non-dimensional temperature gradient in terms of powers of the Peclet number ϵ starting with an ϵ^{-1} term,

$$T(\mathbf{x},\epsilon) \sim \epsilon^{-1}T_{-1}(\mathbf{x}) + T_0(\mathbf{x}) + \epsilon T_1(\mathbf{x}) \quad \text{as} \quad \epsilon \to 0. \tag{3.1}$$

The temperature fields $T_n(\mathbf{x})$ will depend on the configuration of the particles and also on α which for this section is being taken as unity. Substituting the expansion (3.1) into the temperature equation (2.5) yields a sequence of problems

$$\epsilon^{-1}: \qquad 0 = \nabla^2 T_{-1}, \tag{3.2a}$$

$$\epsilon^{0}: \mathbf{u}.\nabla T_{-1} = \nabla^2 T_0 + Q, \tag{3.2b}$$

$$\epsilon^{1}: \quad \mathbf{u}.\nabla T_0 = \nabla^2 T_1. \tag{3.2c}$$

Thus the lowest approximation T_{-1} is a conduction field which we may take to be

$$T_{-1} = Gx \quad \text{everywhere}, \tag{3.3}$$

since a temperature gradient perpendicular to the flow will not interact with the bulk flow which carries away the heat generated by the heat sources. The value of the unknown temperature gradient G is determined from (3.2*b*) which contains the heat sources.

We therefore proceed to examine (3.2*b*), which upon substituting our result (3.3) for T_{-1} becomes

$$\nabla^2 T_0 = u_x G - Q. \tag{3.4}$$

An ensemble average of this equation leads to

$$\nabla^2 \langle T_0 \rangle_0 = G - \phi,$$

because in our non-dimensionalization the bulk average velocity is unity in the x direction, while the heat source strength is unity within the particles and zero outside. Now we must take

$$\nabla^2 \langle T_0 \rangle_0 = 0,$$

since, otherwise, $\langle T_0 \rangle_0$ would grow at least as rapidly as r^2 which will break the asymptotic form of our expansion (3.1). Hence we conclude that

$$G = \phi, \tag{3.5}$$

and thus we have determined the magnitude of the underlying temperature gradient, a result which of course could have also been obtained directly from an overall heat balance. Furthermore, since there are no heat sources in (3.2*c*) and in the equations for higher orders, we see that similar arguments will give

$$\langle T_n \rangle_0 = 0 \quad \text{for} \quad n \geqslant 0.$$

Clearly the leading-order term in the temperature field, $\epsilon^{-1}T_{-1}$, does not render the particles hotter than the bulk, i.e. does not contribute to the integral (2.6), and so we must examine further the next approximation T_0.

3.2. *Average temperature outside a fixed particle*

In the previous sub-section we derived (3.4), the equation governing the temperature field T_0, which was then ensemble averaged to determine the value of G given in (3.5). We now conditionally average (3.4) with one particle fixed at $\mathbf{x}_1$ to obtain an equation governing the average temperature outside a particle,

$$\nabla^2 \langle T_0 \rangle_1 = \langle u_x \rangle_1 \phi - \langle Q \rangle_1. \tag{3.6}$$

With our assumed uniform probability distribution for the separation of particles outside the excluded region $|\mathbf{x}-\mathbf{x}_1| \leqslant 2$ the conditionally averaged heat source $\langle Q \rangle_1$ is given by

$$\langle Q \rangle_1 (\mathbf{x}|\mathbf{x}_1) = \begin{cases} \phi, & r \geqslant 3, \\ \phi(27-56r+30r^2-r^4)/16r, & 1 \leqslant r \leqslant 3, \\ 1, & r \leqslant 1, \end{cases} \tag{3.7}$$

where $r = |\mathbf{x}-\mathbf{x}_1|$. The boundary condition on $\langle T_0 \rangle_1$ is that it should tend to the bulk far from the fixed particle, i.e.

$$\langle T_0 \rangle_1 (\mathbf{x}|\mathbf{x}_1) \to 0 \quad \text{as} \quad r \to \infty.$$

We note that, in the governing equation (3.6), $\langle u_x \rangle_1 \to \langle u_x \rangle_0 = 1$ and $\langle Q \rangle_1 = \phi$ as $r \to \infty$ so that far from the fixed particle the conditionally averaged equation assumes the balance of the bulk equation.

The heat generated within the particle in (3.7) renders this particle hotter on average than the bulk by an amount

$$\Delta T_1 = \tfrac{2}{5} \quad (\alpha = 1).$$

We shall now concern ourselves with the corrections to this expression which results from the $O(\phi)$ terms from outside the fixed particle on the right-hand side of (3.6). As we shall consider these correction terms individually, it is worth calculating first the contribution to ΔT from a general forcing q on the right-hand side of (3.6) which acts only outside the fixed particle. We thus consider

$$\nabla^2 \langle T_0 \rangle_1 (\mathbf{x}|\mathbf{o}) = \begin{cases} q(\mathbf{x}), & r \geqslant 1, \\ 0, & r < 1. \end{cases}$$

Now the calculation of ΔT from $\langle T_0 \rangle_1$ in (2.6) involves an integral over a volume of a sphere. Thus only the spherically symmetric part of $\langle T_0 \rangle_1$ will contribute to ΔT and this part is forced by the spherically symmetric part of q. Let us denote the spherically symmetric part of a quantity by an over-bar, i.e.

$$\bar{q}(r) = \frac{1}{4\pi r^2} \int_{|\mathbf{x}|=r} q(\mathbf{x})\, dA.$$

The equation governing the spherically symmetric part of $\langle T_0\rangle_1$ with the general forcing $q(\mathbf{x})$ is therefore

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\overline{\langle T_0\rangle_1}(r|\mathbf{o})\right) = \bar{q}(r),$$

whose solution satisfying the boundary conditions of regularity at $r = 0$ and decay as $r \to \infty$ is

$$\overline{\langle T_0\rangle_1}(r|\mathbf{o}) = -\frac{1}{r}\int_0^r r'^2\bar{q}(r')\,dr' - \int_r^\infty r'\bar{q}(r')\,dr'.$$

Hence with $\bar{q}(r)$ vanishing inside $r = 1$, we have the contribution to ΔT forced by $q(\mathbf{x})$ as

$$\Delta T = -\int_1^\infty r\bar{q}(r)\,dr. \tag{3.8}$$

With the above result we can readily calculate the contribution to ΔT forced by the varying part, $\langle Q\rangle_1 - \phi$, of the heat source strength in the region $1 \leqslant r \leqslant 3$ given by (3.7). Simple integration yields a contribution

$$\Delta T_2 = -\tfrac{7}{5}\phi.$$

We now turn to the contribution from the velocity disturbance $\langle u_x\rangle_1 - 1$. First we note that if we were to use for this velocity disturbance the expression from the solution of the Stokes equations (2.1) outside an isolated sphere,

$$-\frac{1}{r} + \frac{3x^2 - r^2}{4r^2}\left(\frac{1}{r^3} - \frac{1}{r}\right),$$

our integral (3.8) would clearly diverge. We remarked in §2, however, that in a fixed bed the average of the velocity disturbance satisfies to $O(\phi)$ the Brinkman equation (2.3), whose solution (2.4) gives in (3.8) a contribution

$$\Delta T_3 = \phi\kappa^{-1} = \frac{2^{\frac{1}{2}}}{3}\phi^{\frac{1}{2}} + o(\phi^{\frac{1}{2}}) \quad \text{as} \quad \phi \to 0, \tag{3.9}$$

using a non-dimensionalized $\kappa^2 = \frac{9}{2}\phi$. We see then that the $O(\phi)$ forcing of $\nabla^2\langle T_0\rangle_1$ produces a larger $O(\phi^{\frac{1}{2}})$ term in ΔT because we must integrate out a long distance $r = O(\phi^{-\frac{1}{2}})$ before the integrand in (3.8) changes from its $O(\phi)$ value to an $O(\phi\kappa^{-2}r^{-2})$ decaying function in $r \gtrsim \phi^{-\frac{1}{2}}$. In addition though, since a small $O(\phi)$ term from the $O(1)$ velocity disturbance has produced a larger $O(\phi^{\frac{1}{2}})$ contribution to ΔT, we need to examine carefully the $O(\phi)$ corrections to the average of the velocity disturbance in case they produce unexpectedly larger effects in ΔT.

3.3. *The average velocity disturbance in a fixed bed*

We obtain an exact equation governing $\langle \mathbf{u}\rangle_1(\mathbf{x}|\mathbf{x}_1)$ by conditionally averaging the Stokes equations (2.1) with an appropriate form of the conservation of momentum inside the particles. After some manipulations (see, for example, Hinch 1977) the non-dimensionalized result is

$$\nabla . \langle \mathbf{u}\rangle_1(\mathbf{x}|\mathbf{x}_1) = 0, \quad -\nabla\langle p_f\rangle(\mathbf{x}|\mathbf{x}_1) + \nabla^2\langle \mathbf{u}\rangle_1(\mathbf{x}|\mathbf{x}_1)$$
$$= \int_{|\mathbf{x}_2-\mathbf{x}|=1} \langle \boldsymbol{\sigma}\rangle_2(\mathbf{x}|\mathbf{x}_1,\mathbf{x}_2).\mathbf{n}_2 P(\mathbf{x}_2|\mathbf{x}_1)\,dA_2, \tag{3.10}$$

in which p_f is the pressure in the fluid, $\boldsymbol{\sigma}$ the stress tensor, and $P(\mathbf{x}_2|\mathbf{x}_1)$ is the probability of finding a second particle centred at $\mathbf{x}_2$ given that there is one centred at $\mathbf{x}_1$. (We have assumed earlier that $P(\mathbf{x}_2|\mathbf{x}_1) = 3\phi/4\pi$ if $|\mathbf{x}_2-\mathbf{x}_1| \geqslant 2$ and $P(\mathbf{x}_2|\mathbf{x}_1) = 0$ if $|\mathbf{x}_2-\mathbf{x}_1| < 2$.)

To leading order at large separations, the particle centred at $\mathbf{x}_2$ exerts a dimensionless force -6π in the x direction, corresponding to the dimensional force $-6\pi\mu a\,\mathbf{U}_0$. The first correction to the effect of the $\mathbf{x}_2$-particle is a modification of this force due to the velocity disturbance from the $\mathbf{x}_1$-particle, which changes this force to $-6\pi\langle\mathbf{u}\rangle_1(\mathbf{x}_2|\mathbf{x}_1)$. The next correction is a change in the force exerted by the $\mathbf{x}_2$-particle in response to a similar change to $-6\pi\langle\mathbf{u}\rangle_1(\mathbf{x}_1|\mathbf{x}_2)$ in the force exerted by the $\mathbf{x}_1$-particle – the so-called second reflexion. At higher corrections there is a third reflexion, plus a force $-\pi\nabla^2\langle\mathbf{u}\rangle_1(\mathbf{x}_2|\mathbf{x}_1)$ which arises from the curvature of the flow and is given by Faxen's law, and finally a force dipole reflecting the fact that the fixed bed has an effective viscosity different from that of the pure fluid.

If we add to both sides of the momentum equation (3.10) a force density corresponding to $\mathbf{x}_2$-particles uniformly distributed in $|\mathbf{x}_2-\mathbf{x}_1| \geqslant 1$ exerting at their centres forces $-6\pi\langle\mathbf{u}\rangle_1$, we obtain without approximation Brinkman's equation (2.3), but with a right-hand side forcing

$$-\nabla\langle p_f\rangle_1+\nabla^2\langle\mathbf{u}\rangle_1-\kappa^2\langle\mathbf{u}\rangle_1 = \int_{|\mathbf{x}_2-\mathbf{x}_1|\geqslant 1} dV_2\left\{P(\mathbf{x}_2|\mathbf{x}_1)\int_{|\mathbf{x}'-\mathbf{x}_2|=1}\langle\boldsymbol{\sigma}\rangle_2(\mathbf{x}'|\mathbf{x}_1,\mathbf{x}_2).\mathbf{n}'\right.$$
$$\left.\times\,\delta(\mathbf{x}'-\mathbf{x})\,dA'-\kappa^2\langle\mathbf{u}\rangle_1(\mathbf{x}_2|\mathbf{x}_1)\,\delta(\mathbf{x}_2-\mathbf{x})\right\}, \quad (3.11)$$

where to a first approximation we can take

$$\kappa^2 = \tfrac{9}{2}\phi. \quad (3.12)$$

In the last sub-section we calculated the contribution to ΔT, (3.9), using the velocity disturbance (2.4) which satisfies Brinkman's equation (2.3) and so we ignored the effects of the small $O(\phi)$ term on the right-hand side of (3.11). We shall now consider the velocity disturbance forced by this term. Actually we shall not calculate the whole of the velocity correction, but only that part which when multiplied by ϕ in the temperature equation (3.6) induces an unexpectedly large $O(\phi)$ contribution to ΔT.

Now the solution (2.4) of Brinkman's equation without the right-hand side predicts that the non-dimensional drag on the $\mathbf{x}_1$-particle is increased from the Stokes value of 6π to $6\pi(1+\kappa+\frac{1}{3}\kappa^2)$. In order to take account of the same effect of the fixed bed on the velocity disturbance with two fixed particles, which is used in evaluating $\langle\boldsymbol{\sigma}\rangle_2$ in (3.11), we have to augment the magnitude of the force exerted by the $\mathbf{x}_2$-particle by the same factor in the Brinkman term which we added to the left-hand side of (3.10) to give (3.11). Thus we have to modify expression (3.12) for κ to

$$\kappa^2 = \tfrac{9}{2}\phi(1+\kappa+\tfrac{1}{3}\kappa^2),$$

so that now

$$\kappa = \frac{3}{2^{\frac{1}{2}}}\phi^{\frac{1}{2}}+\tfrac{9}{4}\phi+o(\phi) \quad \text{as} \quad \phi\to 0. \quad (3.12')$$

Interestingly enough the next term for κ is not $O(\phi^{\frac{3}{4}})$ but $O(\phi^{\frac{3}{4}}\ln\phi)$ corresponding to the $\phi\ln\phi$ term in the drag law (2.2) found by Childress (1972). Using this improved value of κ, i.e. (3.12′), in the result (3.9) gives a more accurate expression for the contribution of the Brinkman velocity disturbance to ΔT,

$$\Delta T_3 = \phi\kappa^{-1} = \frac{2^{\frac{1}{2}}}{3}\phi^{\frac{1}{2}} - \tfrac{1}{2}\phi + O(\phi^{\frac{3}{2}}\ln\phi) \quad \text{as} \quad \phi \to 0. \tag{3.9′}$$

After the above adjustment (3.12′) of the magnitude of the Brinkman term, the leading-order term on the right-hand side of (3.11) comes from the second reflexion. Now the solution (2.4) of the Brinkman equation shows that in a fixed bed the velocity disturbance decays like r^{-1} in $\kappa r \lesssim 1$ and like $\kappa^{-2}r^{-3}$ in $\kappa r \gtrsim 1$. Hence the force density on the right-hand side of (3.11) coming from the second reflexion is $O(\phi r^{-2})$ for $\kappa r \lesssim 1$ and $O(\phi\kappa^{-4}r^{-6})$ for $\kappa r \gtrsim 1$. This force density in (3.11) will induce a velocity $O(\phi)$ in $\kappa r \lesssim 1$ and $O(\phi\kappa^{-6}r^{-6})$ in $\kappa r \gtrsim 1$. Such a velocity disturbance in the integral (3.8) yields a contribution to ΔT which is $O(\phi)$. By the same arguments, (1) the third reflexion has a force density $O(\phi r^{-3}, \phi\kappa^{-6}r^{-9})$, velocities $O(\phi r^{-1}, \phi\kappa^{-8}r^{-9})$ and a contribution to ΔT $O(\phi^{\frac{3}{2}})$; (2) the Faxen correction to the force produces a force density $O(\phi r^{-3}, \phi\kappa^{-2}r^{-5})$, velocities $O(\phi r^{-1}, \phi\kappa^{-4}r^{-5})$ and a contribution to ΔT $O(\phi^{\frac{3}{2}})$; and (3) the $O(\phi)$ change in the effective viscosity adds an $O(\phi)$ correction to the basic solution (2.4) and hence an $O(\phi)$ change in its $O(\phi^{\frac{1}{2}})$ contribution to ΔT. To calculate ΔT correct to $O(\phi)$ we can therefore ignore the effects of the several higher corrections and study only the second reflexion. This we shall do in the next subsection.

3.4. *The contribution from the second reflexion*

The solution of the Brinkman equation (2.3) for a point force $\mathbf{F}$ at the origin is (Howells 1974)

$$\mathbf{u} = \frac{\kappa}{4\pi}[\mathbf{F}f(\kappa r) + \mathbf{x}(\mathbf{F}.\mathbf{x})\,g(\kappa r)/r^2],$$

where

$$f(\rho) = [(1+\rho+\rho^2)\,e^{-\rho} - 1]/\rho^3$$

and

$$g(\rho) = -3f(\rho) + 2e^{-\rho}/\rho.$$

In the second reflexion, the second particle at $\mathbf{x}$ exerts a force equal to -6π times the velocity induced at $\mathbf{x}$ by a point force at the origin which has a magnitude equal to -6π times the velocity induced at the origin by a point force at $\mathbf{x}$ with a magnitude -6π in the x direction. Let the unit vector in the x direction be $\mathbf{e}$. Thus the velocity disturbance induced by the second reflexion is the solution of

$$\nabla.\mathbf{u} = 0, \quad -\nabla p + \nabla^2\mathbf{u} - \kappa^2\mathbf{u} = \tfrac{9}{4}\kappa^4[\mathbf{e}f^2 + \mathbf{x}(\mathbf{e}.\mathbf{x})\,(2fg+g^2)/r^2]. \tag{3.13}$$

Now the pressure in (3.13) must be linear in $\mathbf{e}$ and so takes the form

$$p(\mathbf{x}) = \tfrac{9}{4}\kappa^2(\mathbf{e}.\mathbf{x})\,P(r).$$

Taking the divergence of (3.13) and using the continuity equation on $\mathbf{u}$, we obtain an equation for P

$$\frac{d^2P}{dr^2} + \frac{4}{r}\frac{dP}{dr} = -\frac{1}{r}\frac{d}{dr}(f+g)^2 - \frac{2}{r^2}(2fg+g^2),$$

the solution to which, with P regular at the origin and decaying at infinity (corresponding to the absence of an imposed pressure gradient), is

$$P = \int_r^\infty \frac{1}{r'}\left(\tfrac{4}{3}fg + \tfrac{2}{3}g^2\right) dr' - \frac{1}{r^3}\int_0^r r'^2\left(f^2 + \tfrac{2}{3}fg + \tfrac{1}{3}g^2\right) dr'.$$

Substituting the general form of the pressure into the momentum equation (3.13), we have that

$$\nabla^2\mathbf{u} - \kappa^2\mathbf{u} = \tfrac{9}{4}\kappa^4\left[\mathbf{e}(f^2+P) + \mathbf{x}(\mathbf{e}.\mathbf{x})\left(2fg + g^2 + r\frac{dP}{dr}\right)\Big/ r^2\right].$$

Fortunately in order to calculate the contribution to ΔT we do not need the full solution of $\mathbf{u}$ but only the spherically symmetric part $\overline{u}(r)\mathbf{e}$ which satisfies

$$\frac{d^2}{dr^2}(r\overline{u}) - \kappa^2 r\overline{u} = \kappa F,$$

where

$$F = \tfrac{9}{4}r\left[f^2 + \tfrac{2}{3}fg + \tfrac{1}{3}g^2 + P + \tfrac{1}{3}r\frac{dP}{dr}\right],$$

$$= \tfrac{3}{2}r\left[f^2 + \int_r^\infty \frac{1}{r'}(2fg + g^2)\, dr'\right].$$

The solution for $\overline{u}$ which decays at infinity and vanishes on $r = 1$ is

$$\overline{u} = \frac{\kappa^3}{2r}\left[e^{\kappa(2-r)}\int_1^\infty e^{-\kappa r'} F dr' - e^{-\kappa r}\int_1^r e^{\kappa r'} F dr' - e^{\kappa r}\int_r^\infty e^{-\kappa r'} F dr'\right].$$

Substituting this form of the velocity disturbance into the integral (3.8) yields a contribution to ΔT

$$\kappa^2\phi\int_1^\infty (1 - e^{\kappa(1-r)})\, F(r)\, dr,$$

which, on making use of the expression for F and integrating by parts, becomes

$$\Delta T_4 = \tfrac{3}{2}\phi\int_0^\infty \{f^2\rho(1-e^{-\rho}) + (2fg+g^2)\left[(1+\rho)e^{-\rho} - 1 + \tfrac{1}{2}\rho^2\right]/\rho\}\, d\rho,$$

where we have changed the variable of integration to $\rho = \kappa r$ and taken the limit $\kappa \to 0$ thereby introducing an $O(\phi\kappa) = O(\phi^{\frac{3}{2}})$ error. By numerical integration the value of the integral is found to be 0·259.

Collecting together the various contributions to ΔT, i.e. ΔT_{1-4}, we now have the result for equal conductivities ($\alpha = 1$)

$$\Delta T = \tfrac{2}{5} + \tfrac{1}{3}2^{\frac{1}{2}}\phi^{\frac{1}{2}} - 1{\cdot}511\phi + O(\phi^{\frac{3}{2}}\ln\phi), \tag{3.14}$$

when $\epsilon^2 \ll \phi \ll 1$.

4. Slow flows and arbitrary conductivities: $\epsilon^2 \ll \phi$ and arbitrary α

4.1. *The conduction solution*

We now turn to the modifications of § 3 that are needed when the conductivities are not equal. In slow flows the temperature field can still be expanded in powers of the Peclet number ϵ starting with an ϵ^{-1} term as in (3.1) which satisfies the conduction equation

$$\nabla . k\nabla T_{-1} = 0, \tag{4.1}$$

with the non-dimensional conductivity $k = 1$ in the fluid and $k = \alpha$ in the particles. Although the full solution for T_{-1} can no longer remain the simple linear field (3.3) when the conductivities are unequal, the bulk average of T_{-1} must remain linear in order that the flow carries away the bulk heating, i.e.

$$\langle T_{-1}\rangle_0 = Gx. \tag{4.2}$$

Again the value of the temperature gradient G can be found by considering the problem for the next order term T_0. From the overall heat balance, however, we can see here that its value will not change from $G = \phi$ when $\alpha \neq 1$.

In addition to the bulk average of T_{-1} we shall need to know the conditional averages with one and two particles fixed. If we ensemble average (4.1) with one particle fixed at $\mathbf{x}_1$, we obtain

$$\nabla.\alpha\nabla\langle T_{-1}\rangle_1(\mathbf{x}|\mathbf{x}_1) = 0, \quad \text{if} \quad |\mathbf{x}-\mathbf{x}_1| < 1, \tag{4.3a}$$

i.e. if $\mathbf{x}$ lies inside the particle centred at $\mathbf{x}_1$, while outside the fixed particle

$$\nabla^2\langle T_{-1}\rangle_1(\mathbf{x}|\mathbf{x}_1) + \nabla.(\alpha-1)\int_{|\mathbf{x}_2-\mathbf{x}|\leqslant 1} \nabla\langle T_{-1}\rangle_2(\mathbf{x}|\mathbf{x}_1,\mathbf{x}_2)\,P(\mathbf{x}_2|\mathbf{x}_1)\,dV_2 = 0. \tag{4.3b}$$

Now the integral above is $O(G\phi)$ and so to leading order as $\phi \to 0$, $\langle T_{-1}\rangle_1$ is the well-known conduction solution for an isolated sphere in a temperature gradient

$$\langle T_{-1}\rangle_1(\mathbf{x}|\mathbf{x}_1) \sim \begin{cases} Gx\,3/(\alpha+2), & r \leqslant 1, \\ Gx\,[1-(\alpha-1)/(\alpha+2)\,r^3], & r \geqslant 1, \end{cases} \tag{4.4}$$

as $\phi \to 0$ with $r = |\mathbf{x}-\mathbf{x}_1|$. To estimate the errors introduced by neglecting the integral in (4.3b), we must first consider the problem for $\langle T_{-1}\rangle_2$ which has two fixed particles. When an $O(G\phi)$ integral involving $\langle T_{-1}\rangle_3$ is neglected, the equation governing $\langle T_{-1}\rangle_2$ is similar to (4.3) with the particle conductivity inside the two fixed particles and just the fluid conductivity outside. Thus at large separations, $|\mathbf{x}_2-\mathbf{x}_1| \gg 1$, $\langle T_{-1}\rangle_2$ has the $\mathbf{x}_2$-particle behaving as a dipole of strength $O(G(\alpha-1))$ with a correction dipole $O(G(\alpha-1)^2/|\mathbf{x}_2-\mathbf{x}_1|^3)$. In the integral in (4.3b), these dipoles change the conductivity outside the one fixed particle from the fluid value $k = 1$ to an effective value

$$k^* = 1 + 3\phi\,(\alpha-1)/(\alpha+2).$$

Such a change in the conductivity changes the leading-order solution (4.4) for $\langle T_{-1}\rangle_1$ by $O(G\phi\,(\alpha-1)^2)$ inside the one fixed particle and by $O(G\phi\,(\alpha-1)^2/r^2)$ outside. After the leading-order dipoles the $\mathbf{x}_2$-particle has a quadrupole $O(G(\alpha-1)^2/|\mathbf{x}_2-\mathbf{x}_1|^4)$ which produces an $O(G\phi\,(\alpha-1)^2/r^4)$ change in $\langle T_{-1}\rangle_1$ outside the fixed $\mathbf{x}_1$-particle. Hence we may conclude that the temperature disturbance for one fixed particle is correctly given by (4.4) with a relative error $O(\phi\,(\alpha-1))$.

Despite the complexity of the temperature field T_{-1}, it cannot contribute to ΔT when the fixed bed is reflexionally symmetric, which we hereby assume it to be. (The fixed bed of spheres can first lose its reflexional symmetry in the three-particle statistics, which would yield a contribution to ΔT $O(G\epsilon^{-1}\phi^2)$.) Thus, as in the case of equal conductivities, we must proceed to consider the problem for T_0,

$$\mathbf{u}.\nabla T_{-1} = \nabla.k\nabla T_0 + Q. \tag{4.5}$$

When the conductivities are not equal, however, the process of taking ensemble averages of this equation is beset by two difficulties. First, k is discontinuous and must be handled as in (4.3b). Second the function T_{-1} is no longer as simple as in (3.3) and so the nonlinear left-hand side does not reduce effectively to a linear term. We study this second problem in our next subsection.

4.2. *Averages of the nonlinear advection term*

The bulk average of $\mathbf{u}.\nabla T_{-1}$ can be evaluated by a special trick. Specifically, if we denote fluctuations of quantities about their bulk average values by primes, then

$$\langle \mathbf{u}.\nabla T_{-1}\rangle_0 = \langle \mathbf{u}\rangle_0.\nabla\langle T_{-1}\rangle_0 + \nabla.\langle \mathbf{u}' T'_{-1}\rangle_0.$$

Now whereas $\langle T_{-1}\rangle_0$ is linear in x, the local fluctuations of both $\mathbf{u}$ and T_{-1} caused by the particles are statistically homogeneous, and so $\langle \mathbf{u}' T'_{-1}\rangle_0$ does not vary with position. Thus

$$\langle \mathbf{u}.\nabla T_{-1}\rangle_0 = G,$$

and as in § 3 we conclude that $G = \phi$.

On the other hand once we average with some particles held fixed, we lose the spatial homogeneity which formed the basis of the above simplification and hence we require another approach. This alternative method takes advantage of the diluteness of the bed and produces approximations for the averaged quantities which are asymptotic as $\phi \to 0$. First we take a conditional average of $\mathbf{u}.\nabla T_{-1}$ with one fixed particle, and now we use a prime to denote a fluctuation of a quantity about its conditional-average value, so

$$\langle \mathbf{u}.\nabla T'_{-1}\rangle_1 = \langle \mathbf{u}\rangle_1.\nabla\langle T_{-1}\rangle_1 + \langle \mathbf{u}'.\nabla T'_{-1}\rangle_1.$$

In a dilute random bed, fluctuations about the conditional average with one fixed particle are primarily due to the infrequent occurrence of a nearby second particle, i.e. as $\phi \to 0$

$$\langle \mathbf{u}'.\nabla T'_{-1}\rangle_1 \sim \int \langle \mathbf{u} - \langle \mathbf{u}\rangle_1\rangle_2 . \nabla\langle T_{-1} - \langle T_{-1}\rangle_1\rangle_2 P(\mathbf{x}_2|\mathbf{x}_1)\, dV_2. \tag{4.6}$$

This integral converges rapidly as $\mathbf{x}_2 \to \infty$ because $\langle \mathbf{u} - \langle \mathbf{u}\rangle_1\rangle_2$ decays like $|\mathbf{x}_2 - \mathbf{x}_1|^{-1}$ when $\kappa|\mathbf{x}_2 - \mathbf{x}_1| \lesssim 1$ and $\langle T_{-1} - \langle T_{-1}\rangle_1\rangle_2$ decays like $|\mathbf{x}_2 - \mathbf{x}_1|^{-2}$, and so the integral is $O(G\phi)$ as it naively appears. Although this term is small compared with the leading term $O(G)$, it cannot be neglected unless it induces a negligible contribution to ΔT in the integral expression (3.8). Thus we have to consider how rapidly $\langle \mathbf{u}'.\nabla T'_{-1}\rangle_1$ decays as $|\mathbf{x} - \mathbf{x}_1| \to \infty$. First of all we note that, since at large distances from the fixed $\mathbf{x}_1$-particle the main contribution to the integral (4.6) comes from $\mathbf{x}_2$ near $\mathbf{x}$, it would appear that in this region we should use the fluctuations for an isolated $\mathbf{x}_2$-particle with no $\mathbf{x}_1$-particle; but upon integration these fluctuations cancel, because without the $\mathbf{x}_1$-particle they essentially give the term in the bulk average $\langle \mathbf{u}'.\nabla T'_{-1}\rangle_0$ which is known to vanish. Hence $\langle \mathbf{u}'.\nabla T'_{-1}\rangle_1$ does decay as $|\mathbf{x} - \mathbf{x}_1| \to \infty$ and we have to consider the effect of the $\mathbf{x}_1$-particle on the fluctuations around the second particle. Of all the possible first interactions between the two particles, which we have considered, the leading effect is a modification of the velocity disturbance outside the $\mathbf{x}_2$-particle caused by the velocity $O(|\mathbf{x}_2 - \mathbf{x}_1|^{-1}, \kappa^{-2}|\mathbf{x}_2 - \mathbf{x}_1|^{-3})$ induced by the $\mathbf{x}_1$-particle in the neighbourhood of the $\mathbf{x}_2$-particle. When multiplied by the temperature gradient fluctuation outside the $\mathbf{x}_2$-particle $O(G(\alpha - 1)\,|\mathbf{x} - \mathbf{x}_2|^{-3})$ and integrated over $\mathbf{x}_2$, we find

$$\langle \mathbf{u}'.\nabla T'_{-1}\rangle_1 = O(G(\alpha-1)\,\phi|\mathbf{x} - \mathbf{x}_1|^{-1}, G(\alpha - 1)\,\phi\kappa^{-2}|\mathbf{x} - \mathbf{x}_1|^{-3}).$$

This term induces a contribution to ΔT $O(\phi^{\frac{3}{2}}(\alpha - 1))$ and so we can ignore it.

We should record here that the first correction to the estimate (4.6) comes from less frequent occurrences of two particles being nearby the one fixed particle. This correction is

$$\iint \langle \mathbf{u} - \langle \mathbf{u} - \langle \mathbf{u} \rangle_1 \rangle_2 \rangle_3 . \nabla \langle T_{-1} - \langle T_{-1} - \langle T_{-1} \rangle_1 \rangle_2 \rangle_3 P(\mathbf{x}_2, \mathbf{x}_3 | \mathbf{x}_1) \, dV_3 \, dV_2.$$

We expect that similar arguments to the above will show that this correction is $O(G(\alpha-1)\phi^2)$ and that it decays sufficiently rapidly as $|\mathbf{x}-\mathbf{x}_1| \to \infty$ for it to be neglected.

Finally we must note that a similar argument could be applied to the conditional average with two fixed particles, $\langle \mathbf{u} . \nabla T_{-1} \rangle_2$. All that we need to know about this quantity is that it approaches its value G with deviations $O(G|\mathbf{x}-\mathbf{x}_i|^{-1}, G\kappa^{-2}|\mathbf{x}-\mathbf{x}_i|^{-3})$ away from the two fixed particles.

4.3. *Average temperature outside one fixed particle*

We are now in a position to take a conditional average, with one fixed particle, of (4.5) which governs the temperature field T_0. The result is

$$O = \alpha \nabla^2 \langle T_0 \rangle_1 + 1 \quad \text{in} \quad |\mathbf{x}-\mathbf{x}_1| < 1, \tag{4.7a}$$

i.e. inside the fixed particle, and outside

$$\langle \mathbf{u} . \nabla T_{-1} \rangle_1 = \nabla^2 \langle T_0 \rangle_1 + \langle Q^+ \rangle_1 + \langle Q \rangle_1 \quad \text{in} \quad |\mathbf{x}-\mathbf{x}_1| > 1, \tag{4.7b}$$

where

$$\langle Q^+ \rangle_1 = \nabla . (\alpha - 1) \int_{|\mathbf{x}_2-\mathbf{x}|<1} \nabla \langle T_0 \rangle_2 (\mathbf{x}|\mathbf{x}_1, \mathbf{x}_2) P(\mathbf{x}_2|\mathbf{x}_1) \, dV_2.$$

Now the forcing of $\langle T_0 \rangle_1$ inside the particle in (4.7a) produces a contribution to ΔT,

$$\Delta T_1 = (5\alpha + 1)/15\alpha.$$

The remaining forcing is outside the fixed particle, and for its contribution to ΔT we may continue to use the integral expression (3.8) even when the internal conductivity of the fixed particle is no longer unity. Thus $\langle Q \rangle_1 - \phi$ induces a contribution $\Delta T_2 = -\frac{7}{5}\phi$ as in the case of equal conductivities, while the part $\langle \mathbf{u} \rangle_1 . \nabla \langle T_{-1} \rangle_0 - \phi$ of the term $\langle \mathbf{u} . \nabla T_{-1} \rangle_1$ contributes $\Delta T_3 = \frac{1}{3} 2^{\frac{1}{2}} \phi^{\frac{1}{2}} - 0{\cdot}111\phi$ as in §3.3, including the second reflexion. The remaining part of the nonlinear term to be considered is, according to the approximation derived in §4.2,

$$\langle \mathbf{u} \rangle_1 . \nabla (\langle T_{-1} \rangle_1 - \langle T_{-1} \rangle_0).$$

This term vanishes when the conductivities are equal. To evaluate the contribution from this term we can use in the integral (3.8) the approximation (4.4) for $\langle T_{-1} \rangle_1$ and for $\langle \mathbf{u} \rangle_1$ the unshielded Stokes solution, because the integrand decays like r^{-3} in the region $\kappa r \lesssim 1$. The result is $\Delta T_4 = \frac{1}{8}\phi\,(\alpha-1)/(\alpha+2)$.

We now turn to the contribution to ΔT from the new term $\langle Q^+ \rangle_1$ in (4.7b), which is absent when the conductivities are equal. This term involves the average temperature field with two fixed particles $\langle T_0 \rangle_2 (\mathbf{x}|\mathbf{x}_1, \mathbf{x}_2)$ which satisfies

$$O = \alpha \nabla^2 \langle T_0 \rangle_2 + 1 \quad \text{in} \quad |\mathbf{x}-\mathbf{x}_1| < 1 \quad \text{and in} \quad |\mathbf{x}-\mathbf{x}_2| < 1, \tag{4.8}$$

and

$$\langle \mathbf{u} . \nabla T_{-1} \rangle_2 = \nabla^2 \langle T_0 \rangle_2 + \langle Q^+ \rangle_2 + \langle Q \rangle_2 \quad \text{otherwise},$$

where

$$\langle Q^+ \rangle_2 = \nabla . (\alpha - 1) \int_{|\mathbf{x}_3-\mathbf{x}|<1} \nabla \langle T_0 \rangle_3 (\mathbf{x}|\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) P(\mathbf{x}_3|\mathbf{x}_1, \mathbf{x}_2) \, dV_3.$$

Now outside the two fixed particles the terms $\langle \mathbf{u}.\nabla T_{-1}\rangle_2$, $\langle Q^{-1}\rangle_2$ and $\langle Q\rangle_2$ are all $O(\phi)$. By ignoring them we have that at leading order $\langle T_0\rangle_2$ satisfies the problem of two heated spheres isolated in a fluid of conductivity unity. We shall solve this two-sphere conduction problem in the next subsection by the method of multipole expansions. For the present purposes of discussing the problems associated with calculating ΔT and the size of the errors in our calculation, we will denote by $(5\alpha+1)/15\alpha+\mathsf{T}(\mathbf{x}_1, \mathbf{x}_2; \alpha)$ the value obtained by substituting this leading-order approximation for $\langle T_0\rangle_2$ in the place of $\langle T\rangle_1$ in the integral (2.6). Thus T represents how much hotter the $\mathbf{x}_1$-particle would be in the presence of a heated $\mathbf{x}_2$-particle, the two particles being surrounded by a stationary fluid with a conductivity of unity.

At large separations $|\mathbf{x}_2-\mathbf{x}_1| = r \gg 1$, we have by the method of reflexions that

$$\mathsf{T}(\mathbf{x}_1, \mathbf{x}_2; \alpha) = \tfrac{1}{3}\frac{1}{r} - \tfrac{1}{3}\frac{\alpha-1}{\alpha+2}\frac{1}{r^4} + O\left(\frac{\alpha-1}{r^6}\right).$$

When the conductivities of the particles and fluid are equal, we would only have the first term in which the $\mathbf{x}_2$-particle acts as a point heat source. When the conductivities are unequal, there are some higher-order terms. In the first correction, the $\mathbf{x}_2$-particle acts like a dipole in the undisturbed temperature field outside the heated $\mathbf{x}_1$-particle, while in the following correction it acts like a quadrupole.

We now return to (4.7) for $\langle T_0\rangle_1$ to consider the contribution of an $\mathbf{x}_2$-particle. The first effect of an $\mathbf{x}_2$-particle is to act like a heat source and this effect is accounted for precisely by the contribution of the $\mathbf{x}_2$-particle to $\langle Q\rangle_1$. The remaining effect of the $\mathbf{x}_2$-particle must therefore come through its contribution to $\langle Q^+\rangle_1$. From our preceding discussions concerning the form of T we have that $\langle Q^+\rangle_1$ decays like $O(\phi(\alpha-1)r^{-3})$ as the $\mathbf{x}_2$-particle behaves as a dipole. This leading dipole corresponds to the fixed bed having a conductivity $k^* = 1+3\phi(\alpha-1)/(\alpha+2)$. We could incorporate this $O(\phi)$ modification of the effective conductivity into the $\nabla^2\langle T_0\rangle_1$ term in (4.7) by a re-arrangement of the equation similar to that used to transform (3.10) to (3.11). The decay of $\langle Q^+\rangle_1$ is, however, sufficiently rapid for this to be unnecessary, so we can substitute $\langle Q^+\rangle_1$ directly into the integral (3.8) without further manipulations. In the integration each $\mathbf{x}_2$-particle contributes $\mathsf{T}(\mathbf{x}_1, \mathbf{x}_2; \alpha) - \frac{1}{3}r^{-1}$ and thus we have a contribution to ΔT

$$\Delta T_5 = \int_{|\mathbf{x}_2-\mathbf{x}_1|\geqslant 2} \left[\mathsf{T}(\mathbf{x}_1, \mathbf{x}_2; \alpha) - \tfrac{1}{3}\frac{1}{|\mathbf{x}_2-\mathbf{x}_1|}\right] P(\mathbf{x}_2|\mathbf{x}_1)\, dV_2. \tag{4.9}$$

This convergent integral will be computed in the following subsection by the method of multipole expansions.

Finally we have to return to the neglected $O(\phi)$ terms in (4.8). By similar arguments employed in deriving the behaviour of $\langle Q^+\rangle_1$ we can show that the $\langle Q^+\rangle_2$ will yield a contribution $O(\phi^2)$ in ΔT and so can be safely ignored. The other two terms $\langle \mathbf{u}.\nabla T_{-1}\rangle_2$ and $\langle Q\rangle_2$ cancel far from the two fixed particles as they both approach the value ϕ, reflecting of course the bulk heat balance. It is the slow

$$O(\phi|\mathbf{x}-\mathbf{x}_i|^{-1},\ \phi\kappa^{-2}|\mathbf{x}-\mathbf{x}_i|^{-3})$$

approach of $\langle \mathbf{u}.\nabla T_{-1}\rangle_2$ to this eventual value which is potentially troublesome. This slow decay means that through $\langle \mathbf{u}.\nabla T_{-1}\rangle_2$ there is an effective heat source on the left-hand side of (4.8) with a total strength comparable to that inside the particles. While the total heat strength is comparable, we shall now see that a smaller contribution to ΔT is induced, because the effective heat source is distributed over a large volume $O(\phi^{-\frac{3}{2}})$.

First we note that outside the fixed particles the effective heat source $\langle \mathbf{u}.\nabla T_{-1}\rangle_2$ induces a temperature response $\langle T_0\rangle_2$ which is $O(|\mathbf{x}-\mathbf{x}_i|^{-1})$ in $\kappa|\mathbf{x}-\mathbf{x}_i| \gtrsim 1$ and $O(\kappa)$ in $\kappa|\mathbf{x}-\mathbf{x}_i| \lesssim 1$. The associate temperature gradient is $O(|\mathbf{x}-\mathbf{x}_i|^{-2})$ in $\kappa|\mathbf{x}-\mathbf{x}_i| \gtrsim 1$ and $O(\kappa^2)$ in $\kappa|\mathbf{x}-\mathbf{x}_i| \lesssim 1$. Thus in the presence of the $\mathbf{x}_2$-particle the $\mathbf{x}_1$-particle is hotter than the conduction estimate T through a pole term $O(\kappa, r^{-1})$ with a dipole correction $O((\alpha-1)\kappa^2 r^{-2}, (\alpha-1)r^{-4})$, where $r = |\mathbf{x}_2-\mathbf{x}_1|$. Now returning to (4.7), we see that the new pole-like behaviour of the $\mathbf{x}_2$-particle induced by $\langle \mathbf{u}.\nabla T_{-1}\rangle_2$ is accounted for precisely by the $\mathbf{x}_2$-particle contribution to $\langle \mathbf{u}.\nabla T_{-1}\rangle_1$. This contribution was studied in §4.2 where it was shown to be negligible because of a cancellation which occurred in the averaging. The remaining effect induced by $\langle \mathbf{u}.\nabla T_{-1}\rangle_2$ is a correction to $\langle Q^+\rangle_1$ which is $O(\phi(\alpha-1)\kappa^2 r^{-1}, \phi(\alpha-1)r^{-3})$ from the new dipole-like behaviour. This correction induces a contribution to ΔT which is $O(\phi^{\frac{3}{2}}(\alpha-1))$.

4.4. *The two-sphere conduction problem*

We now return to the detailed calculation to $\mathsf{T}(\mathbf{x}_1, \mathbf{x}_2; \alpha)$ which is needed in (4.9) to obtain the contribution ΔT_5. We require the leading-order approximation to $\langle T_0\rangle_2$ which satisfies (4.8) ignoring the small $O(\phi)$ terms outside the two particles. We solve this problem of pure heat conduction outside two heated spheres using the method of twin multipole expansions as set out by Jeffrey (1973). Thus outside the two spheres we express the temperature field in terms of multipoles at the centre of the spheres with unknown amplitudes a_n

$$\langle T_0\rangle_2(\mathbf{x}|\mathbf{x}_1,\mathbf{x}_2) = \sum_{n=0}^{\infty} a_n \frac{(\mathbf{d}.\nabla)^n}{n!}\frac{1}{|\mathbf{x}-\mathbf{x}_1|} + \sum_{n=0}^{\infty} a_n \frac{(-\mathbf{d}.\nabla)^n}{n!}\frac{1}{|\mathbf{x}-\mathbf{x}_2|}$$

in $$|\mathbf{x}-\mathbf{x}_1|,\quad |\mathbf{x}-\mathbf{x}_2| \geqslant 1,$$

where $\mathbf{d} = (\mathbf{x}_1-\mathbf{x}_2)/|\mathbf{x}_1-\mathbf{x}_2|$ is the unit vector in the direction of the line joining the centres of the spheres. On account of the symmetry of the problem we can set the amplitudes for the two particles equal. Inside the $\mathbf{x}_1$-particle we express the temperature field in terms of growing harmonics with unknown amplitudes b_n

$$\langle T_0\rangle_2 = -\frac{|\mathbf{x}-\mathbf{x}_1|^2}{6\alpha} + \sum_{n=0}^{\infty} b_n |\mathbf{x}-\mathbf{x}_1|^{2n+1}\frac{(\mathbf{d}.\nabla)^n}{n!}\frac{1}{|\mathbf{x}-\mathbf{x}_1|} \quad \text{in} \quad |\mathbf{x}-\mathbf{x}_1| \leqslant 1,$$

in which the first term arises from the heating inside the $\mathbf{x}_1$-particle. There is a similar expression for $\langle T_0\rangle_2$ inside the $\mathbf{x}_2$-particle using the same amplitudes b_n on account of the symmetry.

The unknown amplitudes a_n and b_n are determined by applying at the surface of the $\mathbf{x}_1$-particle the boundary conditions of continuity of temperature and of normal flux of heat. To this end we need to express an $\mathbf{x}_2$-multipole in terms of growing harmonics centred on the $\mathbf{x}_1$-particle. It is straightforward to show by differentiating the Taylor series for $|(\mathbf{x}_1-\mathbf{x}_2)+(\mathbf{x}-\mathbf{x}_1)|^{-1}$ that inside $|\mathbf{x}-\mathbf{x}_1| < |\mathbf{x}_1-\mathbf{x}_2|$

$$(-\mathbf{d}.\nabla)^k \frac{1}{|\mathbf{x}-\mathbf{x}_2|} = \sum_{n=0}^{\infty} \frac{(n+k)!}{n!}\frac{|\mathbf{x}-\mathbf{x}_1|^{2n+1}}{|\mathbf{x}_1-\mathbf{x}_2|^{n+k+1}}\frac{(\mathbf{d}.\nabla)^n}{n!}\frac{1}{|\mathbf{x}-\mathbf{x}_1|}.$$

If we now introduce $R = |\mathbf{x}_1 - \mathbf{x}_2|$ for the separation between the two spheres, we obtain from the boundary conditions on $|\mathbf{x}-\mathbf{x}_1| = 1$

$$a_n + \sum_{k=0}^{\infty} a_k \frac{(n+k)!}{n!\,k!}\frac{1}{R^{n+k+1}} = -\frac{\delta_{n0}}{6\alpha} + b_n,$$

and

$$-(n+1)\,a_n + \sum_{k=0}^{\infty} a_k \frac{(n+k)!}{n!\,k!}\frac{1}{R^{n+k+1}} = \alpha\left(-\frac{\delta_{n0}}{3\alpha} + nb_n\right).$$

This infinite system of equations for the amplitudes a_n, which vary with R, can be truncated and tackled directly on a computer. For our purposes of evaluating the integral for ΔT_5 in (4.9) it is more convenient, however, to expand the amplitudes in inverse powers of R,

$$a_k = \sum_{p=n+1}^{\infty} A_{np} R^{-p}.$$

The coefficients A_{np} can then be calculated one at a time from

$$A_{00} = \tfrac{1}{3}, \quad A_{0p} = 0 \quad \text{for} \quad p \geqslant 1,$$

and

$$A_{np} = -\frac{n(\alpha-1)}{n\alpha+n+1}\sum_{k=0}^{[\frac{1}{2}(p-n-1)]}\frac{(n+k)!}{n!\,k!}A_{k(p-n-1-k)} \quad \text{for} \quad p \geqslant n+1,\; n \geqslant 1.$$

(If the A_{np} are known for $n+p \leqslant m$, the above formula gives A_{np} with $n+p = m+1$ and $m+2$.) From the amplitudes a_n thus obtained the b_n amplitudes can be found from

$$b_0 = \frac{2\alpha+1}{6\alpha} + \sum_{k=0}^{\infty} a_k R^{-1-k} \quad \text{and} \quad b_n = -\frac{2n+1}{n\alpha+n+1}a_n \quad \text{for} \quad n \geqslant 1.$$

We can now evaluate the excess temperature for the $\mathbf{x}_1$-particle due to the presence of the $\mathbf{x}_2$-particle

$$\mathsf{T}(\mathbf{x}_1, \mathbf{x}_2; \alpha) = \frac{3}{4\pi}\int_{|\mathbf{x}-\mathbf{x}_1|\leqslant 1} \langle T_0\rangle_2(\mathbf{x}|\mathbf{x}_1, \mathbf{x}_2)\,dV - \frac{5\alpha+1}{15\alpha} = b_0 - \frac{2\alpha+1}{6\alpha}$$
$$= \tfrac{1}{3}R^{-1} + \sum_{n=1}^{\infty}\sum_{p=n+1}^{\infty} A_{np} R^{-n-p-1}.$$

On substituting this result into the integral (4.8) and performing the integration, we find that

$$\Delta T_5 = 3\phi \sum_{n=1}^{\infty}\sum_{p=n+1}^{\infty}\frac{A_{np}}{n+p-2}\left(\tfrac{1}{2}\right)^{n+p-2}.$$

The double sum was found to converge rapidly, three significant figures being given by the terms with $n+p \leqslant 6$ for values of α not too near 0 and ∞.

Collecting together the various contributions to ΔT we now have the result for arbitrary conductivities

$$\Delta T = \frac{5\alpha+1}{15\alpha} + \tfrac{1}{3}2^{\frac{1}{2}}\phi^{\frac{1}{2}} + \phi\left\{-1{\cdot}511 + \tfrac{1}{8}\frac{\alpha-1}{\alpha+2} + 3\sum_{n=1}^{\infty}\sum_{p=n+1}^{\infty}\frac{A_{np}}{n+p-2}\left(\tfrac{1}{2}\right)^{n+p-2}\right\}, \quad (4.10)$$

when $\epsilon^2 \ll \phi \ll 1$. The dependence of the above curly bracket on α is given by figure 1.

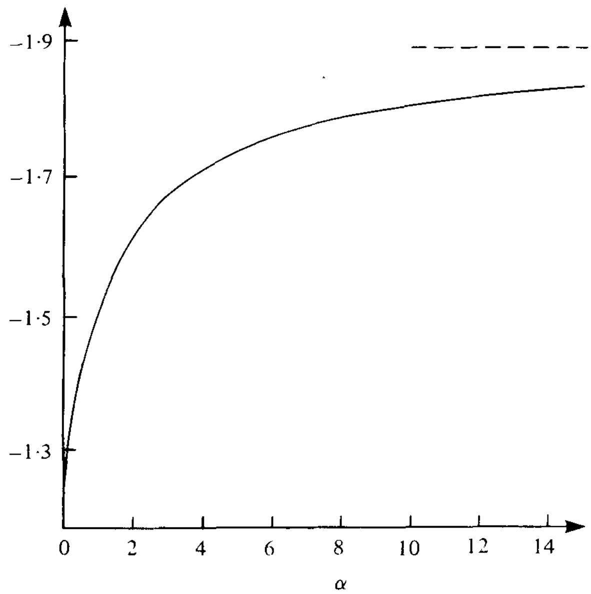

FIGURE 1. The dependence upon the ratio of the conductivities α of the coefficients of the ϕ term in ΔT. The asymptotic result $-1{\cdot}889+0{\cdot}948/\alpha$ is better than $0{\cdot}5\,\%$ accurate when $\alpha > 10$.

5. Comparable advection and shielding: $\phi = O(\epsilon^2)$

When the flow is slow and the fixed bed is very dilute, $\phi \ll \epsilon^2 \ll 1$, Acrivos & Taylor's 1962 analysis of an isolated particle applies. Their result given earlier in (2.7) becomes in our non-dimensionalization

$$\Delta T \sim \frac{5\alpha+1}{15\alpha} - \tfrac{1}{6}\epsilon, \quad \phi \ll \epsilon^2. \tag{5.1a}$$

The $O(\epsilon)$ correction comes from an advection region with a long length scale $r \gtrsim \epsilon^{-1}$ in which the heated particle appears as a point heat source. For this reason the $O(\epsilon)$ correction is independent of α, the particle conductivity. On the other hand, when the fixed bed is dilute and the flow is very slow, $\epsilon^2 \ll \phi \ll 1$, we have just found that

$$\Delta T \sim \frac{5\alpha+1}{15\alpha} + \tfrac{1}{3}2^{\frac{1}{2}}\phi^{\frac{1}{2}}, \quad \epsilon^2 \ll \phi. \tag{5.1b}$$

The $O(\phi^{\frac{1}{2}})$ correction here comes from the hydrodynamically shielded region with the long length scale $r \gtrsim \phi^{-\frac{1}{2}}$. Again the correction is independent of α because the particle appears to be a point in the significant region.

We now explore the transition between the two limiting cases above, i.e. we consider the case $\phi = O(\epsilon^2)$ with $\phi, \epsilon \ll 1$. We can expect a correction $O(\epsilon, \phi^{\frac{1}{2}})$ from a region with a long length scale, and by restricting attention to just this leading-order correction we shall be able to make the following three simplifications: (1) the particle will appear as a point heat source in the region of interest; (2) the approximation (2.4) may be used for the flow, instead of the better results in §3.3; and (3) a number of $O(\phi)$ effects can be ignored.

We start by conditionally averaging with one particle fixed the temperature equation (2.5) suitably non-dimensionalized,

$$O = \nabla . \alpha \nabla \langle T \rangle_1 + 1 \quad \text{in} \quad |\mathbf{x}-\mathbf{x}_1| < 1,$$
$$\epsilon \langle \mathbf{u} . \nabla T \rangle_1 = \nabla^2 \langle T \rangle_1 + \langle Q^+ \rangle_1 + \langle Q \rangle_1 \quad \text{in} \quad |\mathbf{x}-\mathbf{x}_1| > 1,$$

in which $\langle Q \rangle_1$ is given by (3.7) and

$$\langle Q^+ \rangle_1 = \nabla . (\alpha - 1) \int_{|\mathbf{x}_1 - \mathbf{x}| < 1} \nabla \langle T \rangle_2 (\mathbf{x}|\mathbf{x}_1, \mathbf{x}_2) P(\mathbf{x}_2|\mathbf{x}_1) \, dV_2.$$

Note that because we are no longer assuming $\epsilon^2 \ll \phi$ we must not expand the temperature as in (3.1). Now according to § 4.2 we introduce a negligible $O(\phi)$ error by replacing $\langle \mathbf{u} . \nabla T \rangle_1$ by $\langle \mathbf{u} \rangle_1 . \nabla \langle T \rangle_1$. We also know from §§ 4.3 and 3.2 that we can neglect the terms $\langle Q^+ \rangle_1$ and $\langle Q \rangle_1 - \phi$ because they act in a region $r = O(1)$ and produce corrections $O(\phi)$.

We now concentrate on the region in which advection and hydrodynamic shielding are important by introducing the stretched variable $\rho = \epsilon r$. In the region of interest, $\rho = O(1)$, the particle appears to be a point source of heat of strength $\frac{4}{3}\pi$. Also in this region both $\langle T \rangle_1$ and $\langle u \rangle_1$ are within $O(\epsilon)$ of their bulk values, because they both decay like r^{-1} to this region, and so we can write

$$\langle T \rangle_1 = \frac{\phi}{\epsilon} x + \epsilon T' \quad \text{and} \quad \langle \mathbf{u} \rangle_1 = \mathbf{e} + \epsilon \mathbf{u}',$$

where $\mathbf{e}$ is the unit vector in the x direction and T' and $\mathbf{u}'$ are $O(1)$. With all the above approximations and the additional neglect of a small $O(\epsilon)$ term $\mathbf{u}' . \nabla_\rho T'$, our temperature equation has become

$$\mathbf{e} . \nabla_\rho T' - \nabla_\rho^2 T' = \tfrac{4}{3}\pi \, \delta(\rho) - \frac{\phi}{\epsilon^2} \mathbf{e} . \mathbf{u}', \tag{5.2}$$

in which ∇_ρ is the gradient operator in the ρ-scale. Finally making relative errors $O(\phi^{\frac{1}{2}})$ we use expression (2.4) for the velocity disturbance

$$-\frac{\phi}{\epsilon^2} \mathbf{e} . \mathbf{u}' = \tfrac{2}{9} \lambda^2 \left\{ \frac{e^{-\lambda\rho}}{\rho} - \frac{P_2(\cos\theta)}{\lambda^2 \rho^3} [(3 + 3\lambda\rho + \lambda^2\rho^2) \, e^{-\lambda\rho} - 3] \right\},$$

in which we have introduced the parameter $\lambda = \kappa \epsilon^{-1} \sim (\frac{9}{2}\phi)^{\frac{1}{2}} \epsilon^{-1}$ and θ the polar angle from the direction $\mathbf{e}$ of the mean flow.

Now the response in T' to the point heat source on the right-hand side of the governing equation (5.2) is well known to be

$$\tfrac{1}{3}\rho^{-1} \exp\left[\tfrac{1}{2}\rho(\cos\theta - 1)\right].$$

As $\rho \to 0$ this takes the form

$$\tfrac{1}{3}\rho^{-1} + \tfrac{1}{6}(\cos\theta - 1) + O(\rho).$$

Taylor & Acrivos (1962) showed that this form can be matched to an expansion in the region $r = O(1)$ where $\rho = O(\epsilon)$. The $\frac{1}{3}\rho^{-1}$ term matches the conduction solution for a heated particle in a stationary medium, while the $\frac{1}{6}\cos\theta$ term matches to an $O(\epsilon)$ advection correction which has a $\cos\theta$ dependence, and so produces no net contribution to ΔT. Finally the $-\frac{1}{6}$ term produces a uniform cooling in the $r = O(1)$ region with a net contribution to ΔT equal to $-\frac{1}{6}\epsilon$ as given in (5.1*a*).

The velocity disturbance on the right-hand side of (5.2) is $O(\rho^{-1})$ as $\rho \to 0$, and so it forces a response in T' which is regular at $\rho = 0$. We can therefore evaluate the contribution to ΔT by evaluating the forced response of T' at $\rho = 0$. For this calculation we use the Green's function for the left-hand side, which of course is the response to a point heat source given above. The value of T' at $\rho = 0$ forced by the velocity disturbance is thus

$$\tfrac{2}{9}\lambda^2 \int \left\{ \frac{e^{-\lambda\rho}}{\rho} - \frac{P_2(\cos\theta)}{\lambda^2\rho^3} \left[(3+3\lambda\rho+\lambda^2\rho^2)\, e^{-\lambda\rho} - 3 \right] \right\} \frac{e^{\frac{1}{2}\rho(\cos\theta - 1)}}{4\pi\rho}\, dV = \frac{\lambda^2}{3}\left[(1-\lambda^2)\ln\frac{\lambda+1}{\lambda} - \tfrac{1}{2} + \lambda \right].$$

Collecting together the two contributions to ΔT we have when $\lambda^2 = \frac{9}{2}\phi\epsilon^{-2}$ is fixed and $\epsilon \to 0$

$$\Delta T \sim \frac{5\alpha+1}{15\alpha} + \tfrac{1}{3}\epsilon\left[\lambda^2(1-\lambda^2)\ln\frac{\lambda+1}{\lambda} + \tfrac{1}{2}(2\lambda^3 - \lambda^2 - 1) \right]. \tag{5.3}$$

The $O(\epsilon)$ correction is negative when $\lambda < 1$. Here the bulk temperature gradient is unimportant and the flow cools the particles by assisting the conduction to remove the heat. The bulk temperature gradient is, however, important in the more concentrated case of $\lambda > 1$, which has the $O(\epsilon)$ correction positive. Here the fluid near a particle moves slower than the bulk and so spends longer being heated with the consequence that both it and the particle are hotter than the bulk. As $\lambda \to 0$ our result (5.3) becomes

$$\Delta T \sim \frac{5\alpha+1}{15\alpha} + \epsilon\left[-\tfrac{1}{6} + \frac{\lambda^2}{3}\left(\ln\frac{1}{\lambda} - \tfrac{1}{2} \right) \right],$$

while as $\lambda \to \infty$ our result becomes

$$\Delta T \sim \frac{5\alpha+1}{15\alpha} + \epsilon[\tfrac{2}{9}\lambda - \tfrac{1}{4}].$$

The leading-order corrections agree with the limiting results (5.1). The following corrections $(3\phi/4\epsilon)\ln(2\epsilon^2/9e\phi)$ and $-\frac{1}{4}\epsilon$ do not however correspond to the corrections $(2\phi)^{\frac{1}{2}}/3$ and $-\frac{1}{6}\epsilon$ which one would obtain if the small effects for $\phi \ll \epsilon^2$ and $\epsilon^2 \ll \phi$ were simply additive.

This work was supported in part by the National Science Foundation under grant ENG 78-09241 and by a NATO research grant 1442.

REFERENCES

ACRIVOS, A. & TAYLOR, T. D. 1962 *Phys. Fluids* **5**, 387.

AMINZADEH, K., AL TAHA, T. R., CORNISH, A. R. H., KOLANSKY, M. S. & PFEFFER, R. 1974 *Int. J. Heat Mass Transfer* **17**, 1425.

BATCHELOR, G. K. 1974 *Ann. Rev. Fluid Mech.* **6**, 227.

BRINKMAN, M. C. 1947 *Appl. Sci. Res.* **A1**, 27.

CHILDRESS, S. 1972 *J. Chem. Phys.* **56**, 2527.

GUNN, D. J. 1978 *Int. J. Heat Mass. Transfer* **21**, 467.

HINCH, E. J. 1977 *J. Fluid Mech.* **83**, 695.

HOWELLS, I. D. 1974 *J. Fluid Mech.* **65**, 449.

JEFFREY, D. J. 1973 *Proc. Roy. Soc. Lond.* A **335**, 355.

PFEFFER, R. 1964 *Ind. Engng Chem. Fund.* **3**, 380.

PFEFFER, R. & HAPPEL, J. 1964 *A. I. Ch. E. J.* **10**, 605.

ROWE, P. N. 1963 *Int. J. Heat Mass Transfer* **6**, 989.

SHERWOOD, T. K., PIGFORD, R. L. & WILKE, C. R. 1974 *Mass Transfer*. McGraw-Hill.