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The evolution of slender inviscid drops in an axisymmetric straining flow

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The evolution of the shape of a slender inviscid drop in an axisymmetric straining motion is studied at low Reynolds numbers. It is found that the shape equation can be solved by polynominals with time-dependent coefficients. A global stability result can be used to show simply that only one possible equilibrium is stable. It is further shown that if the slender drop starts with a long-wavelength waist then it cannot go to this stable equilibrium and must either extend indefinitely or burst. In the class of trinomial shapes, it is shown that the drop either bursts or goes to the stable equilibrium, depending on whether or not the initial shape has a waist.

1. Introduction

A small drop of a low-viscosity fluid suspended in a fluid of high viscosity has been observed by Taylor (1934) and Torza, Cox & Mason (1972) to deform into a long slender shape when placed in a shearing flow. Taylor (1964) used a slender-body theory for low-Reynolds-number flows to analyse slender drops in an axisymmetric straining motion. He found a slender equilibrium shape for sufficiently strong flows, $E\mu A/\gamma \gg 1$, where E is the principal strain rate, μ the viscosity of the suspending fluid, γ the surface tension and $\frac{4}{3}\pi A^3$ the volume of the drop. Taylor predicted further that drops would break up if the flow was too strong, $E\mu A/\gamma > 0{\cdot}148\,(\mu/\mu_1)^{\frac{1}{6}}$ where μ_1 is the internal viscosity. In this paper we shall consider axisymmetric straining flows which are sufficiently strong for the drop to be slender but not too strong for internal pressure differences proportional to the internal viscosity to be important, i.e. we consider slender inviscid drops with $0{\cdot}148\,(\mu/\mu_1)^{\frac{1}{6}} \gg E\mu A/\gamma \gg 1$.

Buckmaster (1972, 1973) re-examined Taylor's analysis and pointed out that Taylor had missed a wide class of possible solutions to his equation which governed the shape of a steady drop. Buckmaster observed that for a steady drop the cross-sectional radius R could be a function of distance along its axis z according to

$$R = A(\gamma/E\mu A)\,(1-|z|^n/l^n)/2n,$$

with

$$l = A(E\mu A/\gamma)^2\,\tfrac{4}{3}(n+1)\,(2n+1),$$

where n is an arbitrary positive real number; Taylor having selected $n = 2$. Buckmaster thought that it would be desirable if the steady shape were a smooth analytic function, i.e. n were an even integer. This restriction to analytic shapes was found to be necessary by Acrivos & Lo (1978) when they studied a small singular region of the flow near the centre of the drop, see also Brady (1980). In this paper we shall restrict

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attention to time-dependent shapes which are smooth analytic functions and so avoid fast processes which might occur in this small singular region.

The final selection of a single realizable equilibrium shape $n = 2$ (Taylor's choice!) from the infinite number of equilibria with n an even integer was made by Acrivos & Lo (1978) after a stability analysis had found all the other equilibria unstable. In this paper we shall go beyond Acrivos & Lo's infinitesimal stability analysis; we shall consider the general initial value problem for the evolution of the shape.

Some recent studies of slender drops (Hinch & Acrivos 1979, 1980) have developed the theory for two-dimensional straining motion and for simple shear flow. In this paper, however, we shall keep to the simpler axisymmetric straining motion.

The equations governing the evolution of a slender inviscid drop in an axisymmetric flow are derived in the following section. Then in §3 a polynominal in the axial distance with time-dependent coefficients is shown to satisfy this governing equation. The case of a trinomial is studied fully: it is shown that depending on the initial shape, such a drop goes to a unique stable steady shape, or it bursts into two smaller drops in a finite time. Generalizations to an arbitrary polynomial are made in §4. Finally in §5, the positions of local maxima and minima of the radius are shown to move with the undisturbed axial flow. It can then be concluded that a slender drop with a long-wavelength waist cannot go to a steady equilibrium without part of it breaking off.

2. The governing equation

Following Acrivos & Lo (1978), we consider a drop placed in an uniaxial straining motion, given in cylindrical co-ordinates by

$$u_r = -\tfrac{1}{2}Er, \quad u_z = Ez.$$

The drop is chosen to be axially symmetric with a surface at

$$r = R(z,t), \quad l_1 \leqslant z \leqslant l_2.$$

The normal to this surface is

$$n_r = (1+R_z^2)^{-\frac{1}{2}}, \quad n_z = -R_z(1+R_z{}^2)^{-\frac{1}{2}}.$$

We shall assume that the drop is long and slender, i.e.

$$R \ll l_2 - l_1 \quad \text{and} \quad |R_z| \ll 1.$$

Thus we shall not be able to study the evolution of disturbances which have a wavelength which is not long compared with the thickness of the drop. For a slender drop Acrivos & Lo have shown that the Stokes flow outside the drop can be represented, at leading order in the slenderness, by the undisturbed flow plus the flow due to a line distribution of sources $Q(z)$ in $l_1 \leqslant z \leqslant l_2$. Thus for $r = O(R)$ and $l_1 \leqslant z \leqslant l_2$

$$u_r \sim -\tfrac{1}{2}Er + \frac{Q(z)}{2\pi r}, \quad u_z \sim Ez.$$

As there are no pressure gradients at the leading order, the normal stress exerted by the flow on the drop is

$$\sigma_{rr}(z) = -\mu E - \mu Q(z)/\pi R^2(z).$$

This normal stress must balance the interior pressure p of the drop, plus the jump in normal stress due to the surface tension,

$$\sigma_{rr}(z) = -p + \frac{\gamma}{R(z)}.$$

Here we have assumed that the second curvature R_{zz} is negligible compared with R^{-1} for the long slender drop. Thus we have excluded here the stabilizing influence of surface tension on the short wavelength disturbances. We assume further that the viscosity of the drop is so small that the interior pressure p is constant along the drop, the pressure changing in time so as to keep the volume of the drop constant.

Solving the normal stress balance, we obtain the line source strength $Q(z)$ in terms of the instantaneous shape of the drop. This source strength can be substituted into the velocity field. The evolution of the drop shape then follows from the kinematic boundary condition

$$R_t n_r = \mathbf{u}.\mathbf{n} \quad \text{at} \quad r = R.$$

Note on the right-hand side we must retain the small n_z because u_z is $O(l/R)$ larger than u_r. Rearranging, we obtain the governing equation

$$R_t + EzR_z - \left(\frac{P}{2\mu} - E\right) R = -\frac{\gamma}{2\mu}.$$

It is now convenient to non-dimensionalize the problem without change of notation. We scale time with E^{-1} and R with $\gamma/E\mu$. The pressure is scaled with $2\mu E$ and measured with respect to $2\mu E$ so as to absorb the constant $-E$ in the bracket. In order to preserve the volume of the drop as $\frac{4}{3}\pi A^3$ we must scale z, l_1 and l_2 with $A^3E^2\mu^2/\gamma^2$. The condition that the drop is slender is thus the condition of high strain rates:

$$(\gamma/E\mu A)^3 \ll 1.$$

The non-dimensional governing equation is then

$$R_t + zR_z - pR = -\tfrac{1}{2}. \tag{1}$$

This equation is defined over the range

$$l_1(t) \leqslant z \leqslant l_2(t)$$

which varies in time as the drop changes length (drop ends $R(z,t) = 0$ at $z = l_1(t)$ and $l_2(t)$). If the radius of the drop shrinks in time to zero at some point between the ends, the drop will break into two parts. The governing equation would then be applicable to the two parts separately, with possibly different pressures in the two parts.

The pressure p is required to keep the volume of the drop constant in time. In the non-dimensional variables the volume is

$$\int_{l_1}^{l_2} R^2\, dz = \tfrac{4}{3}.$$

Differentiating this constraint with respect to time, substituting R_t from the governing equation, and using $R = 0$ at the ends yields

$$p(t) = -\tfrac{1}{2} + \tfrac{1}{2}\int_{l_1}^{l_2} R\, dz \bigg/ \int_{l_1}^{l_2} R^2\, dz. \tag{2}$$

It is this expression for p which makes the governing equation (1) nonlinear.

3. Some particular solutions

Polynomial shapes. An intriguing property of the shape equation (1) is that it is polynomial-preserving: a polynomial in z with time-dependent coefficients gives a closed solution to (1), because the multiplication by z restores the z derivative to its original order. It is useful first to study solutions with the particular form

$$\mathrm{R}(z,t) = a(t) - b(t)\, z^n - c(t)\, z^m.$$

This form is sufficiently flexible to enable full generalizations to be made in the following section. To avoid examining a certain singular region at $z = 0$, we let m and n be even integers (Acrivos & Lo 1978) with $m > n$. At least one of b and c must be positive, and we shall see later that if one is negative, it cannot be very negative.

Substituting the above polynomial shape into the shape equation (1) yields equations for the development of the coefficients

$$\dot{a} - pa = -\tfrac{1}{2}, \tag{3a}$$

$$\dot{b} + (n-p)\, b = 0, \tag{3b}$$

$$\dot{c} + (m-p)\, c = 0. \tag{3c}$$

This system of equations must be solved with p (a function of a, b and c) found from equation (2), with $l = -l_1 = l_2$ the lowest positive root of the polynomial.

We shall view the solution of equation (3) as a trajectory on the bc plane. This is permitted because a can be found in terms of b and c from the volume constraint. Figure 1 shows the bc-phase plane solution for $n = 2$ and $m = 4$. We argue below that this particular case is typical of the general case.

Figure 1 was obtained numerically by integrating equations (3) with a fourth order Runge–Kutta scheme. It is worth noting that the numerical integration was unstable to volume-changing disturbances if the integral in the denominator in (2) was replaced by the value $\frac{4}{3}$.

Shape-preserving solutions. We first examine shape-preserving solutions, those with b or c vanishing. Suppose $c = 0$ initially, then equation (3c) implies that c vanishes at all times. From the definition of the end, $\mathrm{R} = 0$, we then have $b = al^{-n}$. The normalization of the volume then requires $l = (2n+1)(n+1)/3n^2a^2$. Thus in the case $c = 0$ we can relate b and l to a. The pressure can now be evaluated in equation (2) in terms of a. Substituting this pressure into (3a) yields a particularly simple equation for a with solution

$$a(t) = \frac{1}{2n} + \left(a(0) - \frac{1}{2n}\right) e^{-\frac{1}{2}t}.$$

Thus starting from $c = 0$ and any value of b, we tend along the b-axis in the bc-plane to the point solution

$$B\!: \; b = [3/4(2n+1)(n+1)]^n/2n, \quad c = 0.$$

A similar conclusion applies to c, with $b = 0$ and m replacing n.

Shape changes. An immediate conclusion from (3b) and (3c) is that the signs of b and c do not change in time. Hence the solution trajectory remains in the same quadrant in the bc plane.

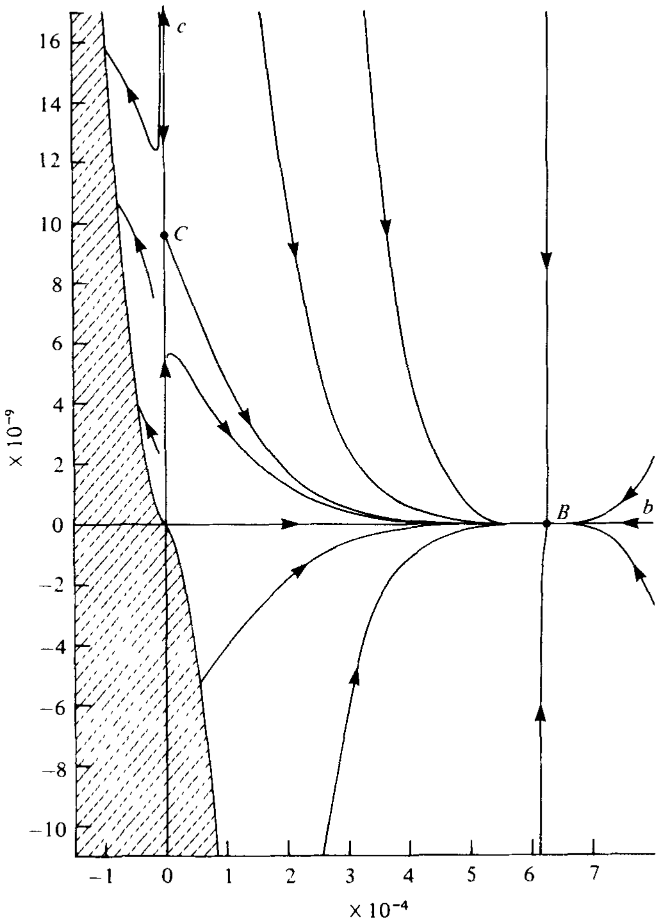

FIGURE 1. Solution trajectories in the bc plane for $n = 2$ and $m = 4$.

A more powerful deduction from (3b) and (3c) is that

$$\frac{b(t)}{c(t)} = \frac{b(0)}{c(0)} e^{(m-n)t}. \tag{4}$$

Hence (with $m > n$) b grows exponentially relative to c, unless b vanishes initially. The acute angle between the b axis and the line from the origin to the point $(b(t), c(t))$ thus rapidly decreases.

A consequence of the above result is that the point solution B on the b-axis is stable to c-perturbations, whereas the point solution C on the c-axis is unstable to b-perturbations. Acrivos & Lo (1978) came to the same conclusions about the stability of the equilibrium shapes but by considering the linearized motion near the equilibria. The above prediction (4) plays the role of a global stability result.

The origin and infinity. If b and c are initially both small, the solution will start rather as if $b = 0$. Thus from the shape-preserving solutions we can expect c to grow, and from equation (4) we must expect b also to grow.

If initially b is small and c is somewhat smaller, $O(b^{(2m+1)/(2n+1)})$, then the solution

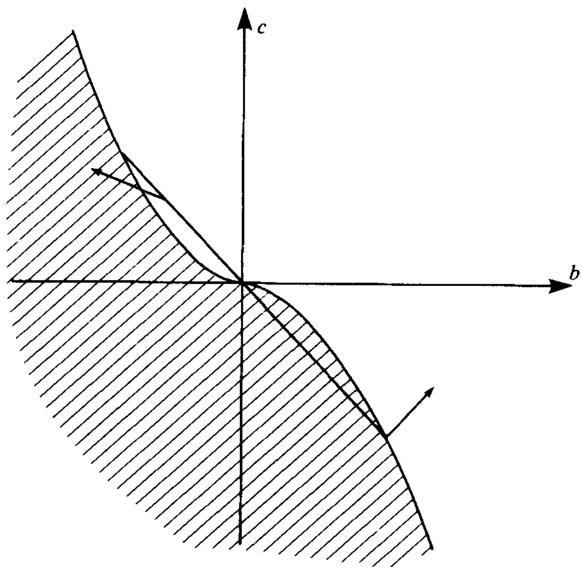

FIGURE 2. The inaccessible regions.

differs from that with $b = 0$. However in this case it is clear from the volume normalization and the definition of the length that a is small $O(b^{1/(2n+1)})$ and l is large $O(b^{-2/(2n+1)})$. Hence in equation (2) p is large $O(b^{-1/(2n+1)})$ and so both b and c grow.

Similar arguments can be presented to show that if b and c are initially large then they must decay.

Inaccessible regions. We must now consider what values of b and c give sensible shapes.

When $b < 0$ and $c > 0$ the drop has a waist, i.e. a constriction at its centre. From physical intuition and from equation (3a), which gives $\dot{a} = -\frac{1}{2}$ at $a = 0$, it is clear that there is a possibility with this shape that in time $a(t) = R(0,t)$ will decrease to zero and thus the drop will burst into two. At $a = 0$ we have $l = (-b/c)^{1/(m-n)}$ and then from the volume normalization

$$c = \left[\frac{3(m-n)^2}{(2m+1)(m+n+1)(2n+1)}\right]^{(m-n)/(2n+1)} (-b)^{(2m+1)/(2n+1)}. \tag{5}$$

For a to be positive b must be greater than the value which gives the equality in (5). The inaccessible region in the bc plane of nonsensical shapes is thus the shaded region in the second quadrant of figure 2.

To find the motion near this forbidden region we note that any straight line from the origin meets the curve (5) once in the quadrant, because the index of the power laws exceeds unity ($m > n$). By the global stability result (4) the solution trajectories move across this line away from the c axis, and so must eventually meet the curve. (They cannot escape to the origin because b and c grow when they are small.) Hence any drop shape with $b < 0$ at the initial instant must burst.

When $b > 0$ and $c < 0$, there is a possibility that the shape of the drop does not close, the limiting case having a cusped end. From the volume normalization and the condition for a double root of $R(z,t) = 0$ at $z = l$, we find

$$c > -k(m,n)\, b^{(2m+1)/(2n+1)}, \tag{6}$$

where $k(m,n)$ is a complicated function of m and n and is positive. We now argue that the drops must move away from this region of inadmissible shapes in the fourth quadrant. First we observe that the global stability result (4) implies a solution trajectory might only enter the region of inadmissible shapes with $\dot{b} < 0$ and $\dot{c} > 0$. But from the expression for the pressure (2) we see $p > -\frac{1}{2}$ and so by (3*b*) and (3*c*) along such a trajectory $dc/db < (2m+1)c/(2n+1)b$. This restriction on the tangent to the trajectory just ensures that the trajectory cannot leave the region (6).

4. Further solutions

In the previous section we considered particular shapes

$$R(z,t) = a(t) - b(t)\, z^n - c(t)\, z^m,$$

m and n even integers, and concluded that if such a drop starts with $b > 0$ it must go to the point solution B, while if it starts with $b < 0$ it must burst by $R(0,t)$ decreasing to zero. This simple solution will now be used to discuss the evolution of an arbitrary shape.

We first consider the more general shape of an arbitrary polynomial of even powers of z,

$$R(z,t) = \sum_{k=0}^{n} a_{2k}(t)\, z^{2k}.$$

If we use the volume constraint to eliminate a_0, we can view the solution as a trajectory in the $a_2 a_4 \ldots a_{2n}$-phase space. The previous section gives the solution on the special two-dimensional surfaces in this phase space which have all but two of the a_{2k} vanishing. In the full phase space there will be a complicated domain of inadmissible shapes.

Now for the more general shape, the equation governing a_0 has the form (3*a*) and the equations governing a_{2k} $(k \geqslant 1)$ have the form (3*b*) and (3*c*). There are thus n point solutions, one on each of the a_{2k} axes $(k \geqslant 1)$, corresponding to steady drop shapes. Only one shape, however, is stable (the one with $a_2 \neq 0$), the remaining ones being unstable to perturbations in shape with a lower degree polynomial.

From the equations governing the a_{2k} $(k \geqslant 1)$ we can deduce that the signs of the a_{2k} do not change. Hence the solution trajectory is confined to a $(\frac{1}{2})^n$ part of the phase space. The equations governing the a_{2k} $(k \geqslant 1)$ also lead to a series of $\frac{1}{2}n(n-1)$ global stability results similar to equation (4), which say that the coefficients of the lower powers grow exponentially relative to the coefficients of the higher powers. Thus the confined trajectory is compelled to move rapidly towards the a_2 axis. The drop must therefore tend to the unique steady shape, or if the trajectory first enters the domain of inadmissible shapes, the drop must burst. If initially the $a_{2k} < 0$ for $1 \leqslant k \leqslant n-1$, the drop will certainly go to the stable steady shape; while if initially $a_2 > 0$, the drop will certainly burst, Further division of the initial conditions does not seem possible with the above approach, but a further criterion is given in §5.

We now consider the further generalization of the polynomial shape to include odd powers of z, e.g. $a_{2k+1}(t)\, z^{2k+1}$. With odd powers included the drop is no longer reflectionally symmetric about $z = 0$. In the extended phase space there are no point solutions corresponding to steady shapes on the a_{2k+1} axes, and in fact the entire a_{2k+1} axes must be in the domain of inadmissible shapes.

If the lowest odd power is higher than a non-trivial even power, then the coefficients of the odd powers all decay exponentially relative to the coefficient of this even power, just like the coefficients of the higher even powers. Thus the significance of the odd powers decays and the drop will go to a steady even shape or burst as before.

If the lowest odd power is lower than all the non-trivial even powers, as must happen for $a_1(t)\, z$, then the coefficient of this lowest odd power will grow exponentially relative to the other coefficients. Although the solution trajectory in the phase spece is rapidly moving away from the even axes on which the equilibria are to be found, the drop need not burst. Instead the drop, which is no longer reflectionally symmetric, can just be translating exponentially fast along the z axis; the exponential change in the translation compensating precisely for the exponential decay of the coefficients of the higher powers, so restoring their significance. This translation can be conveniently absorbed by referring the shape to a moving origin at $z = z_0 e^t$, i.e.

$$R(z,t) = \sum_{k=0}^{n} a_k(t)\, z^k = \sum_{k=0}^{n} a_k'(t)(z - z_0 e^t)^n,$$

where $z = z_0$ is the position of the maximum of $R(z,t)$ at $t = 0$. The equations governing the a_k' differ from those governing the a_k only in the expression for the pressure. As the lowest non-trivial coefficient a_k' is for an even power at $t = 0$ (if $R(z, 0)$ is to have a maximum at $z = z_0$), we may thus deduce the lowest non-trivial coefficient a_k' is always for an even power (and $z = z_0 e^t$ remains a maximum of $R(z,t)$). Thus the case of the lowest non-trivial power being odd has been turned into the opposite even case by referring the shape to a moving origin, and we may conclude again the drop either bursts or goes to a steady shape now translating exponentially along the axis.

5. The motion of stationary points

In § 3 it was shown for a restricted class of polynomial shapes that drops with a single maximum in $R(z, 0)$ went to a steady equilibrium shape, while drops which started with a waist had to burst. Note that the bursting shapes have two maxima and a minimum in $R(z, 0)$. An incidental result at the end of § 4 was that a maximum of $R(z,t)$ moved in time exponentially along the z axis. We now generalize these results. By considering the motion of the stationary points of $R(z,t)$, we will show that if $R(z, 0)$ has more than one maximum the drop cannot go to a steady shape.

Let $z = \zeta(t)$ be a stationary point, i.e. $R_z(\zeta(t), t) = 0$. We can find how $\zeta(t)$ changes in time by differentiating this stationary condition with respect to time

$$\dot{\zeta} = -R_{zt}(\zeta, t)/R_{zz}(\zeta, t).$$

To find R_{zt} we differentiate the governing equation (1) with respect to z,

$$R_{zt} + zR_{zz} + (1-p)\, R_z = 0.$$

Substituting this evaluated at $z = \zeta$, and using the stationary condition again, we obtain

$$\dot{\zeta} = \zeta$$

Thus all stationary points move exponentially along the z axis just as if they are moving with the undisturbed axial flow.

As the stationary points move along the z axis with the same exponential rate, the distance between them must increase in time. Hence the number of stationary points must remain fixed until one of the stationary values of $R(z,t)$ moves in time through zero. This can happen by a minimum value decreasing to zero as the drop bursts, or by a point of inflexion being created or annihilated at the ends.

If we consider the difference between two of the stationary values, $\Delta R = R(\zeta_1(t),t) - R(\zeta_2(t),t)$, then from the governing equation (1) we find

$$\dot{\Delta R} = p\Delta R.$$

The sign of ΔR does not therefore change in time. Hence the relative ordering of all the stationary values cannot change, and so the nature of each stationary point (maximum, minimum or inflexion) is fixed.

We can now conclude that a drop which starts with a shape with two maxima cannot go to a steady shape. The two maxima must be separated by a minimum and the nature of these stationary points is fixed. If the drop does not burst by a minimum value decreasing to zero, the distance between the two maxima must increase exponentially. In neither case does the drop go to a steady shape. The conclusion that a drop cannot go to a steady shape if it has a waist is a criterion that disturbances of the stable equilibrium must exceed a finite amplitude before they destroy the equilibrium.

It seems likely that an arbitrary drop will eventually split up into several fragments, equal in number to the number of maxima in the initial shape, with each fragment tending to a scaled version of the unique stable equilibrium. This however is a conjecture which as yet I cannot support convincingly.

It should be recalled that the conclusions of this section, and the previous sections, are restricted to slender shapes, i.e. those with only long wavelength disturbances.

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