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Longitudinal shear-induced diffusion of spheres in a dilute suspension

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We present a calculation of the hydrodynamic self-diffusion coefficient of a tagged particle in a dilute mono-dispersed suspension of small neutrally buoyant spheres undergoing a steady simple shearing motion. The displacement of the tagged particle parallel to the longitudinal or streamwise direction resulting from a 'collision' with one other particle is calculated on the assumption that inertia and Brownian motion effects are negligible. Summing over different pairs leads to a logarithmically divergent integral for the diffusivity which is rendered finite by allowing for the cut-off due to the occasional presence of another pair of particles. The longitudinal shear-induced self-diffusion coefficient is thus found to be $0.267a^2\gamma\{c \ln c^{-1} + O(c)\}$, where γ denotes the applied shear rate, a is the radius of the spheres and c their volume concentration.

1. Introduction

The shear-induced migration of particles has been shown recently to play an important role in a variety of phenomena involving the flow of concentrated suspensions of solid particles in viscous fluids under conditions in which the flow is laminar and the particle Reynolds number is vanishingly small. For example, the shear-induced particle migration down gradients in concentration is responsible for the phenomenon of viscous re-suspension, wherein a settled layer of heavy particles underlying clear fluid is found to re-suspend under the action of shear (Leighton & Acrivos 1986). Such shear-induced migration in a unidirectional flow can often be represented as a diffusion process with a diffusivity proportional to the local shear rate γ and to the square of the particle radius a, the proportionality factor being a function of the particle concentration c.

The first experimental study of shear-induced diffusion was reported by Eckstein, Bailey & Shapiro (1977) who observed the random lateral motion of a tagged spherical particle in a suspension undergoing shear in a Couette device and thereby computed the coefficient of lateral self-diffusion (in the direction normal to the fluid velocity and the vorticity). Their technique was improved later by Leighton & Acrivos (1987*a*). In addition, by examining carefully the results of several viscometric experiments under a variety of conditions, Leighton & Acrivos (1987*b*) were able to infer values for the diffusion coefficients both within and normal to the plane of shear for the case of particle migration due to gradients in concentration as well as due to gradients in shear. However, aside from numerical simulations in two dimensions (Brady & Bossis 1988), no theoretical calculation of any of these coefficients from basic principles has hitherto been made, mainly owing to the difficulty of analysing the hydrodynamic interaction of more than two particles.

The purpose of the present communication is to derive an expression for the coefficient of shear-induced diffusion *in the direction of the fluid velocity* in a simple shear flow for low values of the concentration c. We focus on this particular diffusivity because it is determined by pairwise particle interactions, which are tractable, in contrast to the transverse diffusivity which requires the solution of a three-sphere hydrodynamic problem. From a practical point of view, knowledge of this longitudinal coefficient is admittedly not of much value, because any lateral displacement of a sphere, resulting say from a three-sphere interaction, leads to differential convection in the streamwise direction which seems likely to dominate the longitudinal displacement. However, the theoretical argument used here may be of some interest as a start on the task of deriving expressions for other and more important shear-induced diffusion coefficients arising in the flow of suspensions.

2. The basic approach

Let us consider a dilute suspension of solid spheres all of radius a which are immersed in a viscous liquid. We suppose that the spheres are neutrally buoyant, that non-hydrodynamic forces are negligible, and that the particle Reynolds number is vanishingly small. We further suppose that the suspension undergoes the simple shearing motion

$$U = (\gamma x_2, 0, 0), \tag{1}$$

where γ is the applied constant shear rate and (x_1, x_2, x_3) is the position vector relative to a fixed origin. All lengths herein are regarded as having been made non-dimensional with the particle radius a. We shall consider in detail only two-sphere interactions. The instantaneous position of the centre of a given sphere, the so-called test sphere, is denoted by X_i and the instantaneous position of the centre of a second sphere is Y_i. The axes are such that the test sphere is stationary when all other spheres are far away. Initially the test sphere is at the origin, i.e. $X_i = 0$, and the second sphere is far away at $(\mp\infty, y_2, y_3)$ where $y_2 \gtrless 0$. It is well known that the resulting two-sphere interaction will not lead to a net lateral displacement of either sphere. In other words, at the end of the encounter the two spheres will be at their initial lateral positions $X_2 = X_3 = 0$, $Y_2 = y_2$, $Y_3 = y_3$. This result follows immediately from the reversibility of the creeping flow equations.

In contrast, there is a non-zero net displacement of the test sphere in the flow direction, as we shall show in §4. The mean value of this displacement, ΔX_1 say, is zero in consequence of the symmetry of encounters with particles coming from the two flow directions. In a dilute dispersion such two-sphere encounters may be regarded as statistically independent, and the standard expression for the self-diffusion, or tracer-diffusion, coefficient of a particle resulting from random displacements with zero mean is then

$$D = \lim_{T\to\infty} \frac{1}{2T} \sum_{k=1}^{N} (\Delta X_1^{(k)})^2, \tag{2}$$

where $\Delta X_1^{(k)}$ is the net displacement of the test sphere in the flow direction resulting from the kth encounter and N is the total number of encounters in time T. The mean rate of occurrence of encounters of the test sphere with spheres of mean number density n having far-upstream lateral coordinates between y_2, y_3 and $y_2+\delta y_2, y_3+\delta y_3$ is $n\gamma\,|y_2|\,\delta y_2\,\delta y_3$, and the displacement of the test sphere resulting from an encounter is a function of y_2 and y_3 alone, whence

$$D = \tfrac{1}{2}n\gamma \int_{-\infty}^{\infty} (\Delta X_1)^2 |y_2|\,\mathrm{d}y_2\,\mathrm{d}y_3. \tag{3}$$

We see that when ΔX_1 is non-zero, two-particle encounters will lead to a shear-induced coefficient of diffusion in the direction of flow which appears to be $O(c)$. In contrast, since $\Delta X_2 = \Delta X_3 = 0$ for any two-sphere interaction, the corresponding diffusivities in each of the two lateral directions would be expected to be $O(c^2)$ and would require for their evaluation the solution of a problem involving the interaction between the test sphere and two other spheres. Although this should be possible in principle, it is a much more formidable task than the one we tackle here.

3. The interaction of two spheres

We denote the vector joining the centres of the two spheres by $r_i \equiv Y_i - X_i$, with r being the centre-to-centre distance. Then, as shown by Batchelor & Green (1972), the relative velocity of the two sphere centres, when the undisturbed velocity is the simple shear flow (1), is given by

$$\frac{\mathrm{d}r_i}{\mathrm{d}t} = \delta_{i1}\gamma r_2 - \tfrac{1}{2}\gamma\,(\delta_{j1} r_2 + \delta_{j2} r_1)\left\{A\frac{r_i r_j}{r^2} + B\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\right\}, \tag{4}$$

where repeated suffixes are summed and the dimensionless scalars A and B are tabulated functions of r with known asymptotic expressions. Moreover, owing to symmetry, the midpoint between the two spheres moves with the undisturbed velocity at that position, whence

$$\frac{\mathrm{d}}{\mathrm{d}t}(X_i + \tfrac{1}{2}r_i) = \delta_{i1}\gamma(X_2 + \tfrac{1}{2}r_2) \quad (i = 1, 2, 3). \tag{5}$$

We deduce that at any time

$$X_2 + \tfrac{1}{2}r_2 = \tfrac{1}{2}y_2, \quad X_3 + \tfrac{1}{2}r_3 = \tfrac{1}{2}y_3, \tag{6}$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}(X_1 + \tfrac{1}{2}r_1) = \tfrac{1}{2}\gamma y_2. \tag{7}$$

From (4) and (7) we find

$$\frac{\mathrm{d}X_1}{\mathrm{d}r_1} = \frac{y_2}{r_2}\,\frac{\tfrac{1}{2}r^2}{r^2 - (A-B)r_1^2 - \tfrac{1}{2}Br^2} - \frac{1}{2}, \tag{8}$$

with $X_1 \to 0$ as $r_1 \to -\infty$ if now without loss of generality we take y_2 to be positive. Thus ΔX_1, the net displacement of the centre of the test sphere at the end of an encounter with a single sphere initially at $(-\infty, y_2, y_3)$, is simply the integral of the right-hand side of (8) with respect to r_1 over the interval $-\infty < r_1 < \infty$. Of course, before this integral can be evaluated, it is necessary to determine the way in which r_2 and r_3 depend on r_1. This can be achieved by solving (4) which, on using r_1 as the independent variable in lieu of the time t, reduces to

$$\frac{r_2\,\mathrm{d}r_2}{(A-B)\,r_2^2 + \tfrac{1}{2}Br^2} = \frac{\mathrm{d}r_3}{(A-B)r_3} = -\frac{r_1\,\mathrm{d}r_1}{r^2 - (A-B)\,r_1^2 - \tfrac{1}{2}Br^2} \tag{9}$$

subject to $r_2 \to y_2$ and $r_3 \to y_3$ as $r_1 \to -\infty$. Thus the task of finding ΔX_1 for given values of y_2 and y_3 reduces to that of solving the system of equations (8) and (9) above.

4. The longitudinal displacement resulting from an encounter

We begin by examining the far-field solution which applies when $y_2 \gg 1$. This requires the use of the following asymptotic forms of A and B as $r \to \infty$:

$$A = \frac{5}{r^3} - \frac{8}{r^5} + O(r^{-6}), \quad B = \frac{16}{3r^5} + o(r^{-6}) \tag{10}$$

given by Batchelor & Green (1972). On substituting these forms into (9) we find by successive approximations that

$$r_2^2 = y_2^2\left(1 + \frac{10}{3r^3} - \frac{16}{3r^5}\right) + \frac{16}{9r^3} + \dots \tag{11}$$

and

$$r_3 = y_3\{1 + O(r^{-3})\}. \tag{12}$$

Hence (8) reduces to

$$\frac{\mathrm{d}X_1}{\mathrm{d}r_1} = \frac{5}{2}\left\{-\frac{1}{3r^3} + \frac{r_1^2}{r^5}\right\} + \left\{\frac{8}{3r^5} - \frac{20r_1^2}{3r^7} - \frac{4}{9y_2^2 r^3}\right\} + \dots \tag{13}$$

in which $r^2 \sim r_1^2 + y_2^2 + y_3^2$. The first term on the right-hand side of (13) arises from the first 'reflection', which affects the motion of the test sphere in two ways. On the one hand, when $y_2 > 0$ the test sphere is pushed downwards into the region of negative fluid velocity. On the other hand, the second sphere pushes the test sphere along the positive 1-axis when $r_1 < 0$ and drags it in the same direction when $r_1 > 0$. It turns out that these two effects cancel each other exactly, as can be seen by integrating the first term of (13) over the interval $-\infty < r_1 < \infty$. We must therefore proceed to the second term in (13) which, when integrated, shows that to leading order

$$\Delta X_1 \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}X_1}{\mathrm{d}r_1}\mathrm{d}r_1 \sim \frac{8}{9}\frac{y_2^2 - y_3^2}{y_2^2(y_2^2+y_3^2)^2} \quad \text{for} \quad y_2 \gg 1. \tag{14}$$

Thus the integrand in (3) decays sufficiently rapidly as y_2 and y_3 approach infinity for the integral to converge.

Next consider the other extreme for which $y_2 \ll 1$. In view of (9), we have to leading order as $r_1 \to \mp\infty$

$$r_2^2 \sim y_2^2 + \frac{16}{9(r_1^2 + y_3^2)^{\frac{3}{2}}}. \tag{15}$$

(Note that negative values of y_2^2 in this equation correspond to the closed trajectories of a sphere pair – see Batchelor & Green 1972. The closed trajectories are interactions between pairs of spheres which persist for an infinite time and which lead to no net displacement of either sphere. Hence we should not include these interactions, and they are excluded by our restriction of the integration to positive values of y_2^2 alone.) In view of (15) we can rearrange (8) into

$$\frac{\mathrm{d}X_1}{\mathrm{d}r_1} \sim \frac{\frac{1}{2}y_2}{\{y_2^2 + \frac{16}{9}(r_1^2+y_3^2)^{-\frac{3}{2}}\}^{\frac{1}{2}}} - \frac{1}{2}, \tag{16}$$

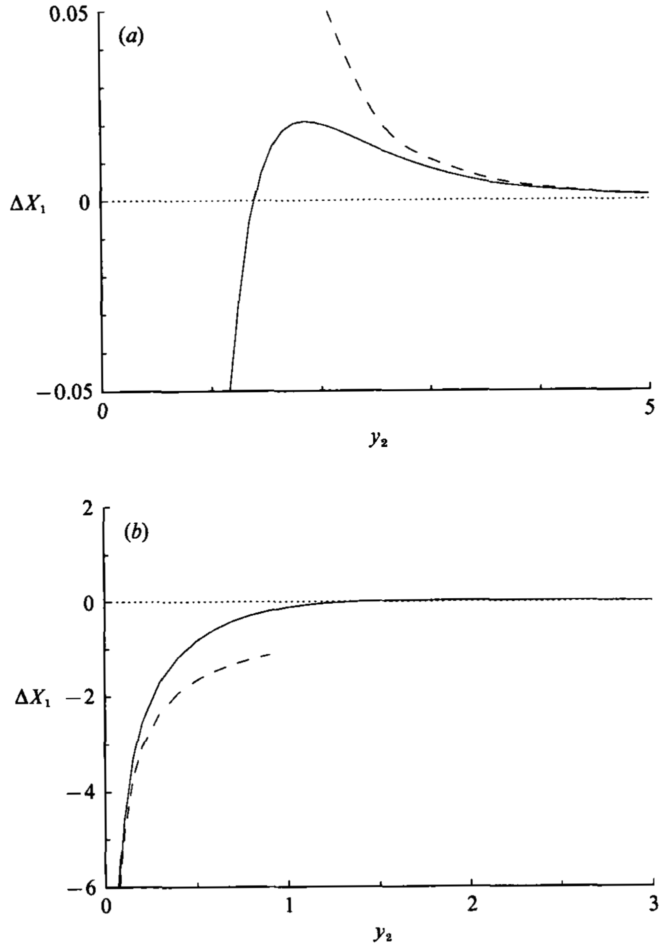

FIGURE 1. The displacement of the test sphere in the direction of flow, ΔX_1, as a function of the upstream offset of a second sphere across the flow in the plane of shear, y_2, with $y_3 = 0$. The continuous curves give the result of a numerical integration of the governing equations. The broken curves give the asymptotic results, (*a*) $\Delta X_1 \sim \frac{8}{9} y_2^{-4}$ as $y_2 \rightarrow \infty$, and (*b*) $\Delta X_1 \sim -1.0477 y_2^{-2/3}$ as $y_2 \rightarrow 0$. Note that the vertical scales in (*a*) and (*b*) are different.

which on integration gives

$$\Delta X_1 \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}X_1}{\mathrm{d}r_1} \mathrm{d}r_1 = -\left(\frac{16}{9y_2^2}\right)^{\frac{1}{3}} I(\beta) \quad \text{for} \quad y_2 \ll 1, \tag{17}$$

where $\beta \equiv (9y_2^2/16)^{\frac{1}{3}} y_3$ and

$$I(\beta) \equiv \int_0^{\infty} [1 - \{1 + (p^2 + \beta^2)^{-\frac{3}{2}}\}^{-\frac{1}{2}}] \,\mathrm{d}p. \tag{18}$$

Although in general the function $I(\beta)$ must be evaluated numerically, we note that $I(0) = \pi^{-\frac{1}{2}} \Gamma(\frac{2}{3}) \Gamma(\frac{5}{6})$, whence

$$\Delta X_1 = -1.0447 y_2^{-\frac{2}{3}} \tag{19}$$

when $y_3 = 0$.

Equation (4) was also solved numerically, and the net longitudinal displacement of the test sphere, ΔX_1, is plotted against y_2 in figure 1 for the case $y_3 = 0$. The numerical results are in excellent agreement with the asymptotic expressions, (14) and (19), in the two extreme ranges of y_2.

5. The determination of the shear-induced coefficient of diffusion

Finally in order to compute the shear-induced diffusion coefficient we return to (3), which requires the evaluation of the double integral. We see from (14) that the integral is absolutely convergent as $y_2 \to \infty$ or $y_3 \to \infty$. However, as $y_2 \to 0$, integration with respect to y_3 gives, in consequence of (17),

$$y_2 \int_0^\infty (\Delta X_1)^2 \,\mathrm{d}y_3 \sim \frac{16}{9y_2}\int_0^\infty I^2(\beta)\,\mathrm{d}\beta = \frac{0.8896}{y_2}. \tag{20}$$

This has a logarithmic singularity when integrated with respect to y_2. With such logarithmic singularities we are fortunate that the leading-order behaviour comes solely from the difference in the orders of magnitude of the small scale of some cut-off (yet to be discussed) and the scale $y_2 = O(1)$; the details on the scale of the cut-off and the details on the scale $y_2 = O(1)$, neither of which will be calculated, contribute corrections smaller by a logarithmic factor.

The singularity arises from very weak but long-lasting interactions between the test sphere and a very slowly moving second sphere when the latter is at great distances from the origin and close to the mid-plane $x_2 = 0$. During such a long interaction, a third particle is likely to pass and interfere with the motion of the original pair. Now a fast third particle will interact separately and reversibly with the two distant spheres, and so will leave their relative configuration virtually unchanged. It is therefore necessary to examine more complicated disruptions of the slow pair. We have considered the following possibilities, finding that (*d*) leads to the largest cut-off of the small y_2: (*a*) the rare interaction of a slow third sphere moving at a similar speed to the slow pair, (*b*) the accumulation from many slightly non-reversible fast third spheres, (*c*) the effect on the test sphere when a third collides with a fourth in its vicinity, (*d*) the effect on the test sphere when a third collides with a fourth at a moderate distance from the test sphere, and (*e*) the accumulative effect on the test sphere from many interactions from distant third and distant fourth particles.

Now, as seen from (15), an approaching second sphere which originates at a small value of y_2 begins to be deflected significantly by the test sphere, i.e. Y_2 changes relative to its initial small value y_2, when $Y_1 = O(y_2^{-\frac{2}{3}})$. The slow interaction of this pair lasts the long time taken to travel the distance Y_1 at the small shear velocity γY_2, i.e. the long time $T = O(\gamma^{-1} y_2^{-\frac{5}{3}})$.

We consider a third sphere colliding with a fourth at a distance L from the test sphere. During the collision the test sphere will feel the effects of the fluctuating dipole exerted by the combined third and fourth spheres. The net effect of this dipole will be to move the test sphere a distance $O(L^{-2})$. Because the system of these three spheres (the test, third, and fourth) is not symmetric, the x_2-component of the net displacement of the test sphere is non-zero and $O(L^{-2})$. (Without the fourth sphere the interaction would have a symmetry which would exclude a net x_2-component.) This interaction will therefore disrupt the slow pair (consisting of the test and second spheres) if this net x_2-component is comparable with the x_2-separation of the slow pair, i.e. if $O(L^{-2}) = O(y_2)$. Hence the slow pair is disrupted by a collision between the third and fourth spheres which are within a distance $L = O(y_2^{-\frac{1}{2}})$ from the test sphere.

Now the rate at which third spheres collide with fourth spheres within the distance L of the given test sphere is $\gamma c^2 L^3$. Multiplying this low rate by the long duration T of the slow pair, we conclude that the slow pair must in all probability be disrupted if

$$1 < O(\gamma c^2 L^3 T) = O(c^2 y_2^{-\frac{3}{2}} y_2^{-\frac{5}{3}}), \quad \text{i.e.} \quad y_2 < O(c^{12/19}).$$

In a similar fashion we can show that the cut-offs for y_2 as obtained by considering the possibilities (*a*), (*c*) and (*e*) mentioned above are respectively $O(c^3)$, $O(c^{\frac{6}{5}})$ and $O(c^{\frac{6}{5}})$, which are all smaller than $O(c^{12/19})$ as $c \rightarrow 0$. The remaining possibility (*b*) is found not to lead to a cut-off.

On summing the contribution to the diffusivity from the slow pairs with

$$c^{12/19} \ll y_2 \ll 1,$$

and using result (20) in (3), we find for the diffusivity in terms of dimensional quantities

$$D = 0.2673a^2\gamma c\,(\ln c^{-1} + O(1)),$$

which is our principal result. In the above, the $O(1)$ correction contains contributions from the fast pair interactions at $y_2 = O(1)$ and contributions from the colliding third and fourth spheres when $y_2 = O(c^{12/19})$. The diffusion process becomes established after the time required for a few slow pair interactions, i.e. a time $T = O(\gamma^{-1}c^{-20/19})$.

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