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Strong streaming induced by a moving thermal wave

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The motion induced in a layer of Boussinesq fluid by a moving thermal wave is studied in the case where the mean velocity could exceed the wave speed. A nonlinear boundary-layer theory shows that strong streaming is possible for small viscosity. Velocity fluctuations are limited in magnitude by their short time scale, while viscosity alone, assumed to be relatively weak, limits the mean flow.

1. Introduction

Following a recent suggestion by Schubert & Whitehead (1969) that the 4-day Venus circulation was produced by periodic thermal forcing there has been considerable interest in understanding the mechanisms by which a moving source of heat imparts an average streaming motion to a fluid. The literature on the phenomenon dates back to Halley (1686) who proposed that periodic heating of the earth's atmosphere by solar radiation might impart a net angular momentum to the atmosphere. The first experimental investigation of the phenomenon was reported by Fultz *et al.* (1959) who rotated a bunsen flame beneath a cylindrical pan of water and generated a weak mean flow in a sense opposite to the rotation of the flame. Stern (1959) performed a similar series of experiments in a cylindrical annulus whose width-to-depth ratio was sufficiently small to reduce the radial convection present in the experiments of Fultz *et al.* By observing paper markers placed on the surface and permanganate tracers in the interior, Stern concluded that a mean rotation was induced in the fluid in a direction opposite to that of the flame. The mean rate of rotation of the water was 0·1 to 1 % of the rotation rate of the source. Stern (1959) also constructed a linear theory to explain his observations and concluded that the shear of the mean motion was supported by the Reynolds stresses of the fluctuating velocity field. The linear theory of the fluid motions induced by a moving thermal wave was extended by Davey (1967) who included the effects of a non-zero thermal diffusivity and a free upper surface boundary condition. More recently, Kelly & Vreeman (1970) have considered the excitation of waves and mean currents in a stratified fluid and have also extended the analysis of Davey for a homogeneous fluid with a free upper surface to the situation in which the thermal wave and surface gravity wave are in resonance.

The moving-flame experiments with water and the theoretical analyses described so far are limited to cases in which the mean velocity of the fluid is much smaller than the source speed. However, large mean currents can be produced by a moving heat source and this is of course the geophysically more interesting case. Schubert & Whitehead (1969) performed a moving-flame experiment with liquid mercury and observed rotation rates at the mercury surface four times as large as the source rotation rate. In experiments performed with periodic internal heating (to be briefly described later) mean velocities comparable to and larger than the phase speed of the internal heating have been observed. The experiments with boundary heating of liquid mercury and internal heating of a salt solution have in common the fact that thermal fluctuations exist within the body of the fluid. In the earlier moving-flame experiments with water the thermal fluctuations were confined to relatively thin boundary regions and as a result only weak mean motions were produced. The generation of large mean flows by periodic thermal forcing is thus a phenomenon of equal *a priori* importance with other modes of circulation in planetary atmospheres. In particular Schubert & Whitehead (1969) have proposed that the retrograde 4-day circulation in the Venus atmosphere can be explained by this phenomenon. The discussion of this proposal at the Fourth Arizona Conference on Planetary Atmospheres, Tucson, Arizona (March 1970) resulted in papers by Schubert & Young (1970) and Malkus (1970).

In this paper we are mainly interested in the phenomenon of strong streaming. After discussing the basic equations and approximations in § 2 we present a non-linear boundary-layer analysis (in § 3) and numerical computations (in § 4) for large induced mean flows. The analysis shows that weak fluctuating motions can indeed produce strong streaming when the viscosity is small. The results of the boundary heating experiments with liquid mercury and the internal heating experiments are briefly discussed in § 5.

2. Basic equations

Consider the two-dimensional motion of a fluid between two rigid horizontal boundaries $z = \pm h$. The motion is induced by a given periodic travelling thermal wave moving horizontally with speed U in the negative x direction. The temperature is independent of vertical position z and is characterized by a wavelength $2\pi/k$. Such a thermal wave could be produced by boundary heating of a fluid of sufficiently small effective Péclet number

$$|U+u|\, kh^2/\kappa \ll 1,$$

where $\mathbf{u} = (u, w)$ is the velocity of the fluid and κ is the thermal diffusivity.

We introduce an average, denoted by an overbar, with respect to the periodicity of the thermal wave and consider only the case of time-independent means. Fluctuating components are denoted by a prime. We assume that the density variations driven by the periodic thermal forcing are sufficiently small for the Boussinesq approximation to be applied (further conditions for the validity of the Boussinesq approximation are given in § 3). The equations expressing conservation of mass for the Boussinesq fluid are

$$\nabla . \mathbf{u}' = 0, \quad \partial \overline{w}/\partial z = 0. \tag{2.1}$$

Thus since $\mathbf{u} = 0$ on $z = \pm h$ the mean vertical velocity $\overline{w} = 0$. The equation describing the variation of the mean horizontal velocity $\overline{u}$ is obtained by averaging the horizontal momentum equation

$$\nu\, d^2\overline{u}/dz^2 = d(\overline{u'w'})/dz, \tag{2.2}$$

where ν is the kinematic viscosity, a constant in the Boussinesq approximation. This equation expresses the balance of the mean flow viscous stress with a Reynolds stress (the vertical divergence of the average vertical transport of horizontal momentum).

We introduce a stream function for the fluctuating velocities according to

$$u' = \partial\psi'/\partial z, \quad w' = -\partial\psi'/\partial x. \tag{2.3}$$

By eliminating the pressure between the horizontal and vertical momentum equations we arrive at a vorticity equation for the stream function

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi' - \frac{\partial\psi'}{\partial x}\frac{d^2\overline{u}}{dz^2} + \left(\frac{\partial\psi'}{\partial z}\nabla^2\frac{\partial\psi'}{\partial x} - \frac{\partial\psi'}{\partial x}\nabla^2\frac{\partial\psi'}{\partial z}\right)' = +\frac{1}{\rho}\frac{\partial\rho'}{\partial x}g + \nu\nabla^4\psi', \tag{2.4}$$

where g is the acceleration of gravity, ρ' is the thermally induced density fluctuation and the density ρ is constant.

It is convenient to introduce non-dimensional variables by using a time scale $(Uk)^{-1}$, a horizontal length scale k^{-1}, a vertical length scale h, and a density fluctuation scale $\Delta\rho$. Thus the horizontal velocity scale is U, the vertical velocity scale is Ukh and the stream-function scale is Uh. The dimensionless forms of (2.2) and (2.4) are

$$\frac{d^2\overline{u}}{dz^2} = S\frac{d}{dz}(\overline{u'w'}), \tag{2.5}$$

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\nabla^2_\beta\psi' - \frac{\partial\psi'}{\partial x}\frac{d^2\overline{u}}{dz^2} + \left(\frac{\partial\psi'}{\partial z}\nabla^2_\beta\frac{\partial\psi'}{\partial x} - \frac{\partial\psi'}{\partial x}\nabla^2_\beta\frac{\partial\psi'}{\partial z}\right)' = F\frac{\partial\rho'}{\partial x} + \frac{1}{S}\nabla^4_\beta\psi', \tag{2.6}$$

where
$$\nabla^2_\beta = \beta^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \tag{2.7}$$

$$F = \frac{\Delta\rho}{\rho}\frac{gh}{U^2}, \quad S = \frac{kUh^2}{\nu} \quad \text{and} \quad \beta = kh. \tag{2.8}$$

We now take the mean field approximation in which the fluctuating part of terms quadratic in the fluctuations is neglected, i.e. the last term on the left-hand side of (2.6) is assumed to be negligible. Clearly a necessary condition for the validity of this approximation is

$$|\psi'| \ll 1. \tag{2.9}$$

In the limiting cases of dominant viscosity $S \ll 1$ and of weak viscosity $S \gg 1$, both of which will be studied in some detail, we find that the mean field approximation is a valid simplification of (2.6) if, in addition to (2.9), we require that any asymmetry in the thermal forcing be smaller than either the expansion parameter S or $S^{-\frac{1}{2}}$. The mean field equation for the stream function is

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\nabla^2_\beta\psi' - \frac{\partial\psi'}{\partial x}\frac{d^2\overline{u}}{dz^2} = F\frac{\partial\rho'}{\partial x} + \frac{1}{S}\nabla^4_\beta\psi', \tag{2.10}$$

subject to the boundary conditions

$$\partial\psi'/\partial x = \partial\psi'/\partial z = 0 \quad \text{on} \quad z = \pm 1. \tag{2.11}$$

The non-linear problem for $\overline{u}$ and ψ' can be Fourier analyzed in (x, t). At present we study only one Fourier component, i.e.

$$\rho', \psi' = \mathrm{Re}\,(\hat{\rho}, \hat{\psi}(z) \exp i(x+t)),$$

where Re denotes the real part of a complex quantity. Equations (2.5) and (2.10) written in terms of the Fourier amplitudes are

$$\frac{d^2\overline{u}}{dz^2} = S\frac{d}{dz}\left\{\tfrac{1}{2}\,\mathrm{Im}\left(\hat{\psi}\,\frac{d\hat{\psi}^*}{dz}\right)\right\}, \tag{2.12}$$

$$\left\{1+\overline{u}+\frac{i}{S}\left(\frac{d^2}{dz^2}-\beta^2\right)\right\}\left(\frac{d^2}{dz^2}-\beta^2\right)\hat{\psi}-\hat{\psi}\,\frac{d^2\overline{u}}{dz^2} = F\hat{\rho}, \tag{2.13}$$

where Im denotes the imaginary part of a complex quantity and the asterisk denotes a complex conjugate. The boundary conditions are

$$\overline{u} = \hat{\psi} = d\hat{\psi}/dz = 0 \quad \text{on} \quad z = \pm 1. \tag{2.14}$$

For algebraic simplicity we will consider the limit $\beta^2 \to 0$. This is a non-singular limit and consequently will not alter the structure of the solutions.

3. Theoretical solutions

In this section we reproduce the results obtained by Stern (1959)† for weak streaming $\overline{u} \ll 1$ in two limiting cases, that of dominant viscosity $S \ll 1$ and weak viscosity $S \gg 1$. For the case of weak viscosity and within the conditions for the validity of equation (2.13), Stern's result (3.18) indicates the possibility of strong streaming. We solve the non-linear problem in the case of weak viscosity and find that the mean velocity profile is the same as in the linearized analysis (to first order).

Without any loss of generality we take $\hat{\rho}$ to be unity, i.e. the density variations are

$$\rho' = \cos(x+t).$$

Following Stern we consider the linearized weak streaming problem

$$\frac{d^2\overline{u}}{dz^2} = S\frac{d}{dz}\left\{\tfrac{1}{2}\,\mathrm{Im}\left(\hat{\psi}\,\frac{d\hat{\psi}^*}{dz}\right)\right\}, \tag{3.1}$$

$$\left(1+\frac{i}{S}\frac{d^2}{dz^2}\right)\frac{d^2\hat{\psi}}{dz^2} = F, \tag{3.2}$$

$$\overline{u} = \hat{\psi} = d\hat{\psi}/dz = 0 \quad \text{on} \quad z = \pm 1. \tag{3.3}$$

Since $\hat{\psi}$ is the solution of a linear constant-coefficient ordinary differential equation it can be written in terms of simple functions. Then through equation (3.1) it is possible, although quite tedious, to write down a closed-form solution

† Except for a minor algebraic correction.

for $\bar{u}$ at arbitrary S. In view of our later extension to the non-linear problem, we elect only to consider two limiting cases of extreme S where it is possible to expand the solution in powers of a small parameter (S or $S^{-\frac{1}{2}}$) and solve simpler approximate equations.

For the dominant viscosity case $S \ll 1$, we write formally

$$\hat{\psi} = S\hat{\psi}_1 + S^2\hat{\psi}_2 + \ldots.$$

The problem for the first approximation is

$$\left.\begin{aligned} d^4\hat{\psi}_1/dz^4 &= -iF, \\ \hat{\psi}_1 = d\hat{\psi}_1/dz &= 0 \quad \text{on} \quad z = \pm 1, \end{aligned}\right\} \tag{3.4}$$

with the solution
$$\hat{\psi}_1 = -(iF/4!)\,(1-z^2)^2. \tag{3.5}$$

Substituting this into (3.1) yields no Reynolds stress because to a first approximation u' and w' are in quadrature. It is therefore necessary to consider the problem for $\hat{\psi}_2$

$$\left.\begin{aligned} d^4\hat{\psi}_2/dz^4 &= i\, d^2\hat{\psi}_1/dz^2 = -(F/3!)\,(1-3z^2), \\ \hat{\psi}_2 = d\hat{\psi}_2/dz &= 0 \quad \text{on} \quad z = \pm 1, \end{aligned}\right\} \tag{3.6}$$

with the solution
$$\hat{\psi}_2 = (F/6!)\,(1-z^2)^2\,(z^2-3). \tag{3.7}$$

From (3.1), (3.5) and (3.7) we find that the first approximation to the mean flow is
$$\bar{u} = (F^2S^4/2\,.\,5!\,6!)\,[1-z^2]^5 + O(F^2S^6). \tag{3.8}$$

The $O(F^2S^5)$ contribution to $\bar{u}$ is identically zero since $\hat{\psi}_3$ is pure imaginary. Condition (2.9) becomes
$$FS \ll 1$$

and thus the form of (3.8) implies that strong streaming is not possible while the mean field equations model the Navier–Stokes equation in the limit of dominant viscosity.

For weak viscosity the expansion in powers of $S^{-\frac{1}{2}}$ is not regular; there are boundary layers of thickness $S^{-\frac{1}{2}}$ at $z = \pm 1$. It is therefore necessary to proceed by the method of matched asymptotic expansions. The boundary-layer equation at $z = -1$ is, in the stretched variable $\zeta = S^{\frac{1}{2}}(z+1)$, exactly

$$\left(1 + i\frac{d^2}{d\zeta^2}\right)\frac{d^2\hat{\psi}}{d\zeta^2} = \frac{F}{S}, \tag{3.9}$$

subject to the boundary conditions

$$\hat{\psi} = d\hat{\psi}/d\zeta = 0 \quad \text{at} \quad \zeta = 0, \tag{3.10}$$

and matching conditions with the as yet undetermined outer solution as $\zeta \to \infty$. The general solution of (3.9) and (3.10) which is well behaved as $\zeta \to \infty$ is

$$\hat{\psi} = A\left[\exp\left\{-\frac{1+i}{2^{\frac{1}{2}}}\zeta\right\} - 1 + \frac{1+i}{2^{\frac{1}{2}}}\zeta\right] + \frac{F}{2S}\zeta^2. \tag{3.11}$$

If we expand the interior solution formally as

$$\hat{\psi} = \hat{\psi}_0 + S^{-\frac{1}{2}}\hat{\psi}_1 + \ldots,$$

then (3.11) and a similar expression for the boundary layer at $z = +1$ yield via the matching a series of constraints on the interior solution

$$\hat{\psi}_0 = 0, \quad \hat{\psi}_1 = \pm\frac{1-i}{2^{\frac{1}{2}}}\frac{d\hat{\psi}_0}{dz} \quad \text{on} \quad z = \pm 1. \tag{3.12}$$

The small (first-order) boundary-layer efflux is driven by the oscillating zeroth-order tangential flow just outside the boundary layer.

The problem for the first approximation to the interior solution is thus

$$d^2\hat{\psi}_0/dz^2 = F,$$

$$\hat{\psi}_0 = 0 \quad \text{at} \quad z = \pm 1, \tag{3.13}$$

with the solution

$$\hat{\psi}_0 = \tfrac{1}{2}F(z^2-1). \tag{3.14}$$

To lowest order in the interior the upward vertical velocity of the travelling convection cell lags the lightest density by $\frac{1}{2}\pi$. As in the viscosity-dominated case, to the first approximation there is no Reynolds stress in the interior. Thus it is necessary to consider the problem for $\hat{\psi}_1$

$$d^2\hat{\psi}_1/dz^2 = 0,$$

$$\hat{\psi}_1 = \pm\frac{1-i}{2^{\frac{1}{2}}}\frac{d\hat{\psi}_0}{dz} = F\frac{1-i}{2^{\frac{1}{2}}} \quad \text{at} \quad z = \pm 1. \tag{3.15}$$

The solution is

$$\hat{\psi}_1 = F(1-i)/2^{\frac{1}{2}}. \tag{3.16}$$

This secondary interior flow is $\frac{1}{4}\pi$ out of phase with the primary flow.

Substituting both (3.14) and (3.16) into (3.1) yields the Reynolds stress

$$d^2\bar{u}_0/dz^2 = -F^2S^{\frac{1}{2}}/2^{\frac{3}{2}}. \tag{3.17}$$

An integration of (3.1) through the boundary layers using the appropriate forms of the stream function yields a matching condition on the interior expansion for $\bar{u}$

$$\bar{u} = O(F^2) \quad \text{at} \quad z = \pm 1.$$

Thus the mean velocity for the first approximation $\bar{u}_0$ vanishes at the matching point. Accordingly the mean velocity profile is

$$\bar{u} = (F^2S^{\frac{1}{2}}/2^{\frac{5}{2}})(1-z^2) + O(F^2). \tag{3.18}$$

Condition (2.9) reduces to

$$F \ll 1,$$

and so, with F fixed and small, and as $S \to \infty$, it seems possible that the mean flow could become greater than the thermal wave speed, although the preceding analysis would then not be applicable. The reason that the quadratic terms in the fluctuations can give rise to large mean effects, but negligible fluctuating corrections, lies in the different structure of the equations they satisfy. In the small diffusion limit the fluctuating part is balanced by a time rate of change of a correction. The mean part, being independent of time, is balanced only by diffusion of the mean response. As the diffusion coefficient is assumed small, the magnitude of the mean quantity has to be compensatingly large.

We are therefore led to a study in which (3.2) is replaced by (2.13) in which terms involving $\bar{u}$ are retained,

$$\left(1+\bar{u}+\frac{i}{S}\frac{d^2}{dz^2}\right)\frac{d^2\hat{\psi}}{dz^2}-\hat{\psi}\frac{d^2\bar{u}}{dz^2}=F. \tag{3.19}$$

Since we are interested only in the limit of weak viscosity, in order to produce strong streaming, we can again employ the method of matched asymptotic expansions. First we examine the boundary layers. Guided by the previous analysis we can estimate the relative importance within the boundary layers of the terms in (3.19). With $\bar{u}=O(F^2)$, $\hat{\psi}=O(FS^{-\frac{1}{2}})$, the length scale $O(S^{-\frac{1}{2}})$ and assuming $F \ll 1$, the boundary layers are linear to the first approximation, satisfying (3.9) as before. Formally expanding the interior as

$$\hat{\psi}=\hat{\psi}_0+S^{-\frac{1}{2}}\hat{\psi}_1+\ldots,$$

we find the same matching conditions (3.12).

When the streaming is strong it is no longer possible to obtain individual equations for $\hat{\psi}_0$ and $\hat{\psi}_1$ since they are coupled by the non-linearity. Thus as the first approximation for the streaming we must consider the problem

$$(1+\bar{u}_0)\frac{d^2\hat{\psi}_0}{dz^2}-\hat{\psi}_0\frac{d^2\bar{u}_0}{dz^2}=F, \tag{3.20}$$

$$(1+\bar{u}_0)\frac{d^2\hat{\psi}_1}{dz^2}-\hat{\psi}_1\frac{d^2\bar{u}_0}{dz^2}=0, \tag{3.21}$$

$$\frac{d^2\bar{u}_0}{dz^2}=\frac{S^{\frac{1}{2}}}{2}\frac{d}{dz}\left\{(\operatorname{Im}\hat{\psi}_1)\frac{d\hat{\psi}_0}{dz}-\hat{\psi}_0\operatorname{Im}\left(\frac{d\hat{\psi}_1}{dz}\right)\right\}, \tag{3.22}$$

subject to the matching conditions

$$\hat{\psi}_0=0,\quad \bar{u}_0=0,\quad \hat{\psi}_1=\pm[(1-i)/2^{\frac{1}{2}}]\,d\hat{\psi}_0/dz \quad \text{at} \quad z=\pm 1. \tag{3.23}$$

To avoid cumbersome non-linear layers in the interior we have chosen to restrict the magnitude of $\bar{u}$, and therefore $F^2S^{\frac{1}{2}}$, to be $O(1)$ and not $O(S^{\frac{1}{2}})$ in the limiting process $S\rightarrow\infty$.

Equation (3.21) can be integrated twice to yield

$$\hat{\psi}_1=C(1+\bar{u}_0),$$

where C is one of the constants of integration, the other vanishing from the symmetry of the problem. The constant C can be determined by integrating (3.20) once. In (3.22) it is possible to eliminate $\hat{\psi}_0$ using (3.20) and (3.21),

$$(1+\bar{u}_0)\,d^2\bar{u}_0/dz^2=\tfrac{1}{2}FS^{\frac{1}{2}}\operatorname{Im}\hat{\psi}_1.$$

The constant C is such that (3.17) is again the governing equation of the mean flow and it is a simple matter to obtain

$$\hat{\psi}_0=\tfrac{1}{2}F(z^2-1), \tag{3.24}$$

$$\hat{\psi}_1=F[(1-i)/2^{\frac{1}{2}}]\,[1+(F^2S^{\frac{1}{2}}/2^{\frac{5}{2}})(1-z^2)], \tag{3.25}$$

$$\bar{u}_0=(F^2S^{\frac{1}{2}}/2^{\frac{5}{2}})(1-z^2). \tag{3.26}$$

We note that the first approximations to the mean flow and the fluctuating stream function are unchanged from their values in the linear analysis. This is fortuitous, however, and would not be so if the top surface were stress free, for example.

When the streaming is strong, $S^{-\frac{1}{2}} \lesssim F^2$, it is not clear that condition (2.9) is sufficient to ensure that the mean field approximation is a valid representation of the Navier–Stokes equations. This is because a possible correction to $\hat{\psi}$ of $O(F^2)$ from the non-linear terms could *a priori* produce a streaming of $O(F^3S)$, which is more important than that produced by the boundary-layer efflux of $O(F^2S^{\frac{1}{2}})$. For a thermal wave of the form

$$\rho' = \mathrm{Re}\sum_{n=1}^{N} \hat{\rho}_n\, e^{in(x+t)}, \tag{3.27}$$

the interior solution

$$\psi' = \mathrm{Re}\sum_{k,m,n=1,0,1}^{K,M,N'} F^k S^{-\frac{1}{2}m}\hat{\psi}_{k,m,n}(z)\, e^{in(x+t)}, \tag{3.28}$$

$$\overline{u} = S\sum_{k,m=2,0}^{K',M'} F^k S^{-\frac{1}{2}m}\overline{u}_{k,m}(z), \tag{3.29}$$

is consistent with the Navier–Stokes equations (2.5) and (2.6) (the limit $\beta^2 \to 0$ will be taken for convenience). A more detailed study of the boundary layers yields the matching conditions

$$\hat{\psi}_{k,0,n} = \overline{u}_{2,0} = \overline{u}_{2,1} = 0, \quad \hat{\psi}_{1,1,n} = \pm[(1-i)/2^{\frac{1}{2}}n^{\frac{1}{2}}]\, d\hat{\psi}_{1,0,n}/dz \quad \text{at} \quad z = \pm 1.$$

If the $\hat{\rho}_n$ are real, i.e. if the thermal wave is symmetric about some phase, then $\hat{\psi}_{1,0,n}$ are real. If the $\hat{\psi}_{k,0,n}$ were real, i.e. cosinusoidal, then the quadratic non-linear term will excite sum and difference harmonics which are also cosinusoidal. Hence by induction over the k, all the $\hat{\psi}_{k,0,n}$ are real and it is impossible for them to contribute to $\overline{u}$ as their velocity fields u' and w' are in quadrature. Thus the first approximation to $\overline{u}$ is indeed $\overline{u}_{2,1}$ from the contributions of the $\hat{\psi}_{1,0,n}$ and $\hat{\psi}_{1,1,n}$. The mean flow is found to be

$$\overline{u} = \frac{F^2S^{\frac{1}{2}}}{2^{\frac{5}{2}}}(1-z^2)\sum_{n=1}^{N}\hat{\rho}_n^2\, n^{\frac{1}{2}} + O(F^3S^{\frac{1}{2}}, F^2).$$

We conclude this section by considering the conditions for the validity of the Boussinesq approximation. These conditions follow from the fact that for the Boussinesq approximation to be valid

$$\left|\frac{1}{\rho}\frac{D\rho}{Dt}\frac{1}{ku'}\right| = \left|\frac{\nabla\,.\,\mathbf{u}'}{ku'}\right| \ll 1,$$

where D/Dt is the convective derivative. For $S \ll 1$ the condition is

$$(gh/U^2)\, S \gg 1,$$

while for $S^{\frac{1}{2}} \gg 1$ the condition becomes simply

$$gh/U^2 \gg 1.$$

When the streaming is strong the condition must be strengthened to

$$(gh/U^2)\,(1+\overline{u})^{-2} \gg 1$$

and the requirement of symmetry of ρ'.

4. Numerical solutions

For intermediate values of viscosity the asymptotic techniques of the previous section must be replaced by numerical ones. In this section we present numerical solutions of the non-linear two-point boundary-value problem represented by equations (2.12) and (2.13) (in the limit $\beta^2 \to 0$). These equations can be integrated to give

$$\frac{d\overline{u}}{dz} = \tfrac{1}{2}S\,\mathrm{Im}\left(\hat{\psi}\,\frac{d\hat{\psi}^*}{dz}\right), \tag{4.1}$$

$$\left(1+\overline{u}+\frac{i}{S}\frac{d^2}{dz^2}\right)\frac{d\hat{\psi}}{dz}-\hat{\psi}\,\frac{d\overline{u}}{dz} = F\hat{\rho}z. \tag{4.2}$$

The constants of integration are zero since $\overline{u}$ and $\hat{\psi}$ are even functions of z.

A forward numerical integration of (4.1) and (4.2) from $z = -1$ to $z = 0$ requires that we know the values of $\overline{u}$, $\hat{\psi}$, $d\hat{\psi}/dz$ and $d^2\hat{\psi}/dz^2$ at $z = -1$. At the lower wall $z = -1$, $\overline{u} = \hat{\psi} = d\hat{\psi}/dz = 0$, and $d^2\hat{\psi}/dz^2$ is adjusted in an iterative manner until the forward integration gives the result $d\hat{\psi}/dz = 0$ at $z = 0$. Equation (4.1) ensures that the slope of the mean velocity profile is zero at the channel centre-line.

The solution of the linearized version of (4.2) provides a starting-point for the entire procedure. For values of F and S consistent with the linear approximation, the analytic solution of the linearized form of equation (4.2) gives the value of $d^2\hat{\psi}/dz^2$ at $z = -1$. Either or both of the parameters can then be changed and the value of the second derivative appropriate to the prior set of parameters can be used to start the numerical procedure for the new set. The process is continued by using the result for the second derivative in a given case to start the iterative scheme for the next set of parameters. In this manner we proceed from the linear into the non-linear régime.

To illustrate the character of the solutions at intermediate values of S we discuss in detail the case $F = 0{\cdot}2$. The dimensionless mean velocity $\overline{u}$ is shown as a function of S in figure 1 together with the results of the two asymptotic theories. There is no difficulty in extending the numerical solutions to sufficiently low values of S that essentially exact agreement with the asymptotic solution is obtained. As we have previously discussed, and as can be seen in figure 1, large mean flows correspond to large values of S. An essential difficulty is encountered in extending the numerical solutions to larger values of S as a result of the fact that the flow tends toward a boundary-layer structure as S becomes large. At $S = 250$, for example, a value of $d^2\hat{\psi}/dz^2$ at $z = -1$ which is inaccurate by as much as $0{\cdot}05\,\%$ will, upon forward integration from $z = -1$, lead to an unbounded result. At $S = 500$ a discrepancy in $d^2\hat{\psi}/dz^2$ $(z = -1)$ in the fifth significant figure leads to a singular result. Thus it was necessary to use double precision accuracy on the IBM 360/91 to obtain the numerical solutions at the relatively larger values of S. This numerical difficulty is a result not of the nonlinear terms in the equations but of the boundary-layer character of the solution.

To give a clear illustration of the nature of the boundary-layer solutions we have chosen the case of $F = 0{\cdot}2$ and $S = 250$. For these values the thickness of

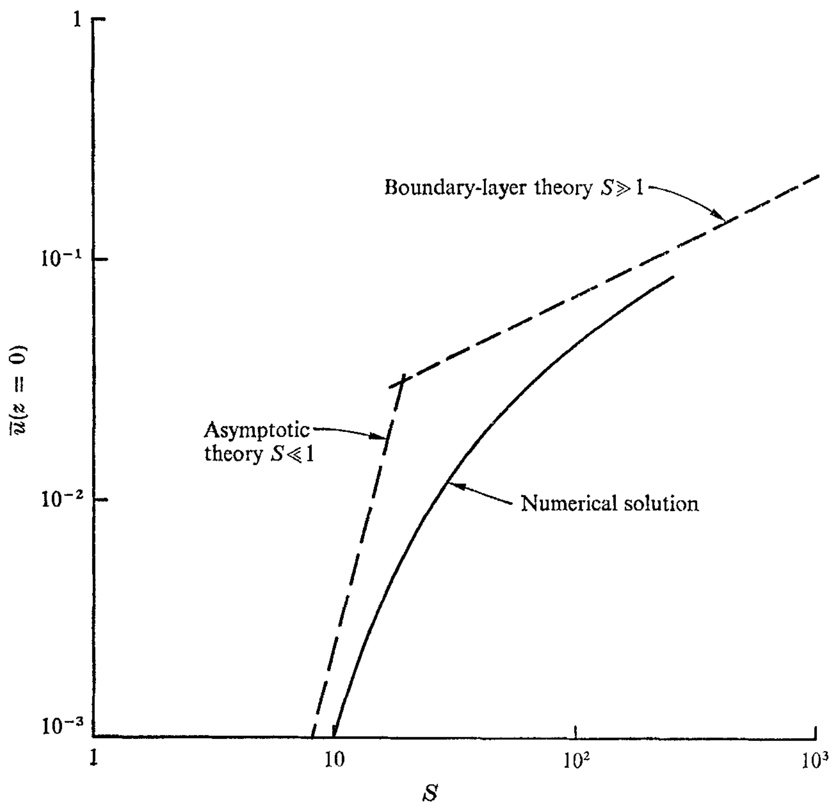

FIGURE 1. Comparison of the numerically determined mean flow with asymptotic results, for $F = 0{\cdot}2$.

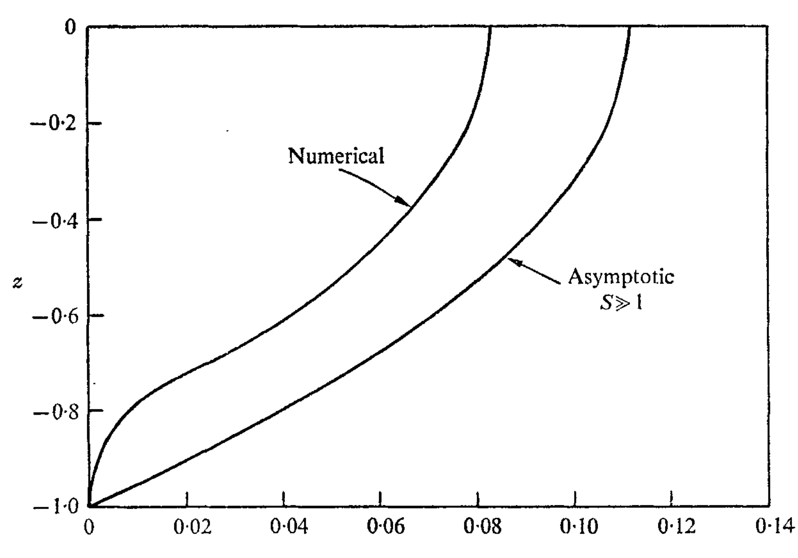

FIGURE 2. Mean velocity profile for $F = 0{\cdot}2$ and $S = 250$.

the boundary layers, arbitrarily defined by when $\exp[-(z+1)(S/2)^{\frac{1}{2}}]$ has dropped to the 5% level, is 0·3 (e.g. see figure 2). The mean velocity $\overline{u}$ is shown as a function of z with the asymptotic result for the interior, equation (3.26), also

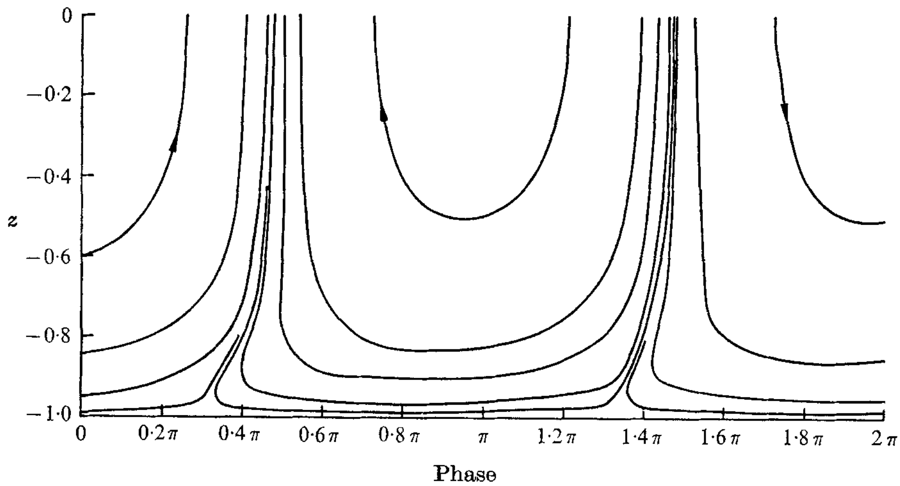

FIGURE 3. The perturbation stream curves for $F = 0{\cdot}2$ and $S = 250$, showing the tilt in the convection cell.

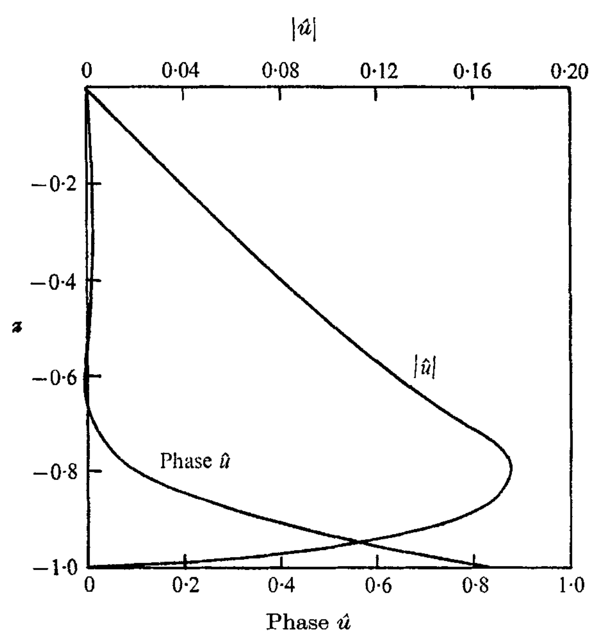

FIGURE 4. The amplitude and phase of the horizontal velocity fluctuations at $F = 0{\cdot}2$ and $S = 250$.

shown for comparison. While the imposed thermal fluctuation moves to the left, the mean motion of the fluid is everywhere to the right. The slope of $\overline{u}$ is zero both at the channel centre and at the walls and an inflexion point in the mean velocity profile occurs at about $z = \pm 0{\cdot}7$. In the steady state there is no shear stress at the walls.

The perturbation stream curves for $S = 250$ are shown in figure 3. It is important to note the tilting of the convection cells which is most readily apparent near the walls where viscosity is relatively more significant. The tilting of the cells is a result of the correlation of positive (negative) vertical velocity fluctuations with

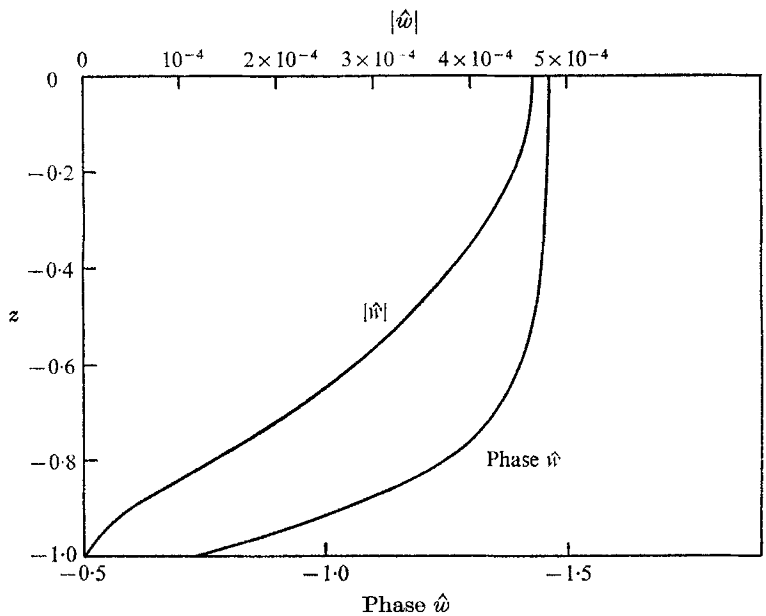

FIGURE 5. The amplitude and phase of the vertical velocity fluctuations at $F = 0{\cdot}2$ and $S = 250$.

positive (negative) horizontal velocity fluctuations. This leads to a transport of positive horizontal momentum toward the channel centre-line which is necessary to support the shear of the mean flow. Finally, for $S = 250$, figures 4 and 5 show the magnitude and phase of the horizontal and vertical velocity fluctuations, respectively. While the horizontal velocities $\hat{u}$ and $\bar{u}$ are independent of β, the magnitude of the vertical velocity $\hat{w}$ is directly proportional to β (the value $\beta = 10^{-2}$ has been used in figure 5). As indicated by the asymptotic theory for large S, over most of the channel interior $\hat{u}$ is approximately in phase with the thermal forcing while $\hat{w}$ is almost $\frac{1}{2}\pi$ radians out of phase with the thermal wave.

5. Strong streaming experiments

In the moving flame experiment with boundary heating of liquid mercury Schubert & Whitehead (1969) have reported mean velocities four times as large as the flame speed and in the opposite direction. Whitehead (1970) has described these experiments in detail, so that our only concern here is with the dependence of the mean dimensionless speed at the upper surface of the mercury on the parameters F and S. His data is summarized in figure 6. It is clear, for the experiments reported in this figure, that the magnitude of the mean flow in the mercury is proportional to F^2S^4, i.e. the flow is viscosity dominated. The values of S in these experiments ranged from 0·53 to 8·8 and F was generally in the range

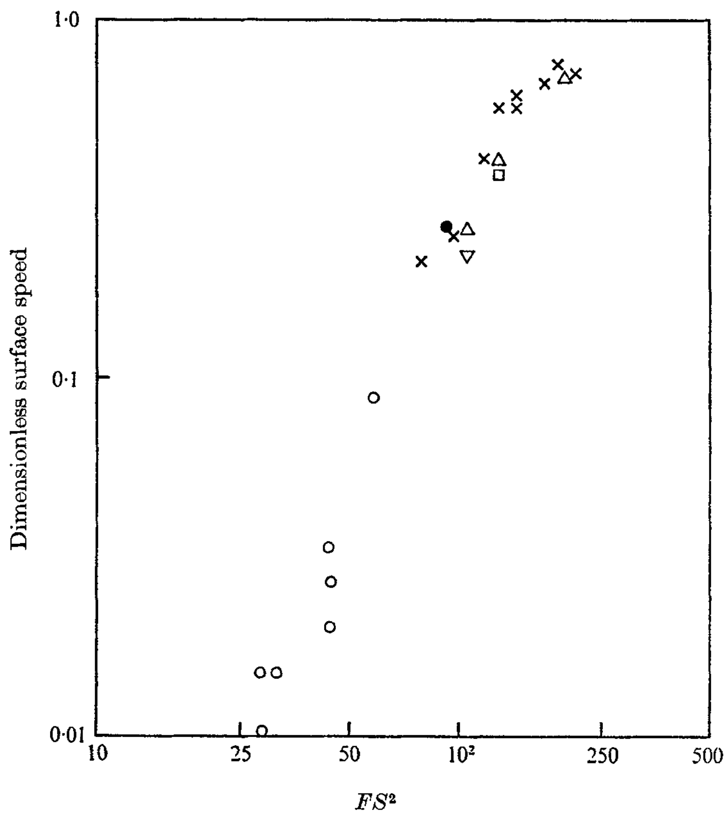

FIGURE 6. The experimentally observed surface speed versus FS^2, for different S: ○, 8·8; ●, 4·8; ▽, 3·17; ×, 1·6; △, 1·1; □, 0·53.

10–100. Although FS^2 is rather large in the experiments and the theory of the viscosity-dominated flows discussed here is strictly valid only for $F \ll 1$, $S \ll 1$, the applicability of the analysis extends beyond these strict limits as a result of the factorials in the denominator of (3.8) arising from the character of the biharmonic operator.

Some simple preliminary 'moving flame' experiments with internal heating, conceived and designed by P. J. Mason and H. A. Douglas, have been performed at the British Meteorological Office, Bracknell. Water in an annular gap (radii 30 mm, 85 mm and depth 110 mm) was heated internally by the ohmic dissipation (up to 1 kVA) of a current passed across the gap. The conductor on the outer cylinder was segmented. By the use of a cam-operated switch, it was possible to make the thermal wave propagate around the annulus at speeds between 1 mm/s and 1 m/s resulting in a variation in S from 1 to 10^3. Mean surface flows, visualized by floating aluminium flakes, of up to twice the thermal wave speed and in the opposite direction were noted. It is hoped that in some future communication more detailed experimental results will be available.

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