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# Steady long slender droplets in two-dimensional straining motion

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The recent analysis by Acrivos & Lo (1978) concerning the breakup of a long slender droplet in an axisymmetric straining motion is extended to the case of a two-dimensional hyperbolic flow. It is found that, although the cross-section of the droplet becomes significantly non-circular, the theoretical criterion for breakup is effectively the same as in the axisymmetric case. The theoretical predictions are in good agreement with the available experimental results.

## 1. Introduction

The deformation and breakup of single droplets freely suspended in another fluid undergoing shear under conditions of creeping flow constitute a subject of longstanding importance in fluid mechanics. According to the experimental evidence, cf. Taylor (1934) and Torza, Cox & Mason (1972), the droplet remains almost spherical, even up to the point of breakup, when its viscosity is comparable to or higher than that of the surrounding fluid, and hence the theoretical analysis can be conveniently performed in these cases via a regular perturbation expansion in the parameter $\epsilon$ which measures the deformation of the droplet shape from spherical (Taylor 1932; Cox 1969; Barthès-Biesel & Acrivos 1973).

Taylor (1934) observed in his four-roller apparatus, however, that droplets became long and slender at high strain rates if their viscosity was much smaller than that of the suspending fluid. This experimental observation has been confirmed by Grace (1971), Torza *et al.* (1972) and Yu (1974). Grace and Yu also found experimentally that the droplet would break up if the strain rate was too high, specifically, when $E\mu a/\gamma > 0{\cdot}12(\mu/\mu_i)^{0{\cdot}16}$, in which $E$ is the strain rate, $\mu$ the viscosity of the suspending fluid, $\mu_i$ the viscosity of the droplet ($\mu_i \ll \mu$), $\frac{4}{3}\pi a^3$ the volume of the droplet and $\gamma$ the surface tension.

Taylor (1964) was the first to propose that the behaviour of these long droplets could be described quantitatively by slender-body theory. He studied droplets in axisymmetric straining flow, a simpler flow than the two-dimensional straining motion of the four-roller apparatus, and predicted that the droplets would break up if $E\mu a/\gamma > 0{\cdot}148(\mu/\mu_i)^{\frac{1}{6}}$. Taylor's analysis has been rendered more rigorous by Buckmaster (1972, 1973) and by Acrivos & Lo (1978). Buckmaster found that there were many possible steady shapes of the droplet, of which all but one were shown to be unstable by Acrivos & Lo (1978). These recent studies were, like Taylor's, confined

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to the axisymmetric case; however, as shown by Acrivos & Lo (1978), their technique can easily be extended to include inertia effects and, in principle, general three-dimensional flows.

In this paper we return to two-dimensional straining motion, where a direct comparison can be made between the theoretical result to be developed and the experimental data. We split the two-dimensional strain into an axisymmetric strain plus a remainder, the axisymmetric strain having the same principal rate of stretching, i.e. we set the velocity gradient tensor equal to

$$\nabla \mathbf{u} = E\begin{pmatrix} -1 & & \\ & 0 & \\ & & 1 \end{pmatrix} = E\begin{pmatrix} -\frac{1}{2} & & \\ & -\frac{1}{2} & \\ & & 1 \end{pmatrix} + Ee\begin{pmatrix} -\frac{1}{2} & & \\ & \frac{1}{2} & \\ & & 0 \end{pmatrix} \quad \text{with} \quad e = 1.$$

The two-dimensional straining motion is then treated as a slightly perturbed axisymmetric strain by means of an expansion in the small parameter $e$. The results are then evaluated by setting the expansion parameter $e$ equal to 1, which is not particularly small, although it will appear that $e$ becomes large in some sense only at $e = 5$.

## 2. The governing equations

We consider a droplet placed in an undisturbed general straining motion given in cylindrical polar co-ordinates $(r, \theta, z)$ by

$$u_r = -\tfrac{1}{2}E(1+e\cos 2\theta)\,r, \quad u_\theta = \tfrac{1}{2}Ee\sin 2\theta r, \quad u_z = Ez.$$

When $e = 0$ this flow is axisymmetric, while when $e = 1$ the flow is a two-dimensional straining motion. The viscous stress field associated with the general straining motion is

$$\sigma_{rr} = -\mu E(1+e\cos 2\theta), \quad \sigma_{r\theta} = \mu Ee\sin 2\theta, \quad \sigma_{\theta\theta} = -\mu E(1-e\cos 2\theta), \quad \sigma_{zz} = 2\mu E,$$

where $\mu$ is the viscosity of the suspending fluid.

We take the droplet surface to be described by

$$r = R(\theta, z) \quad \text{in} \quad |z| \leqslant l.$$

In this paper we shall study only steady shapes which are symmetric about $z = 0$. The normal to the droplet surface is

$$(n_r, n_\theta, n_z) = \left(1, -\frac{\partial R}{\partial \theta}\frac{1}{R}, -\frac{\partial R}{\partial z}\right)\Big/\left(1+\left(\frac{\partial R}{\partial \theta}\right)^2\frac{1}{R^2}+\left(\frac{\partial R}{\partial z}\right)^2\right)^{\frac{1}{2}},$$

and the kinematic boundary condition gives

$$u_r - u_\theta\frac{\partial R}{\partial \theta}\frac{1}{R} - u_z\frac{\partial R}{\partial z} = 0 \quad \text{on} \quad r = R.$$

We seek a solution to the Stokes equations inside and outside the droplet which at infinity will tend to the undisturbed flow. In addition to the kinematic boundary condition on the droplet surface, we require that the velocity be continuous and that the jump in the surface stress be along the unit normal and proportional to the average curvature of the surface $K$, i.e. that

$$[\boldsymbol{\sigma}.\mathbf{n}] = \mathbf{n}\gamma K \quad \text{at} \quad r = R,$$

where $\gamma$ is the surface tension.

We now assume that the droplet is long and slender when $\mu_i$, the viscosity of the droplet, is much smaller than that of the suspending fluid, i.e. we assume that

$$R \ll l, \quad |\partial R/\partial z| \ll 1.$$

The average curvature of the surface then involves only the shape of the bubble in the $r, \theta$ plane, i.e.

$$K = \left(R^2 + 2\left(\frac{\partial R}{\partial \theta}\right)^2 - R\frac{\partial^2 R}{\partial \theta^2}\right) \Bigg/ \left(R^2 + \left(\frac{\partial R}{\partial \theta}\right)^2\right)^{\frac{3}{2}}.$$

In the corresponding axisymmetric case, Acrivos & Lo (1978) have shown that, outside the droplet, the flow remains effectively undisturbed except within a region of thickness $O(R)$, the so-called inner region, where a disturbance flow sets in which is radially symmetric. Also, within the drop, the flow is primarily along the axial ($z$) direction and requires an axial variation in the pressure $p$. The present case is, of course, quite similar to that studied by Acrivos & Lo (1978), hence we shall follow many of the steps of their analyses, which they have already justified in considerable detail.

We can take it for granted then that, within the inner region, the flow outside a slender droplet with a low viscosity is disturbed at leading order in only the $r, \theta$ plane, the axial ($z$) flow being unchanged. [Disturbances to the axial flow are negligible because the bubble can exert only small axial forces when it has a small interior viscosity, whereas disturbances in the $r, \theta$ plane are generated by the forces normal to the surface of the bubble and these are significant because the interior pressure is much larger than the deviatoric stresses.] We can thus substitute $u_z = Ez$ in the kinematic boundary condition. Note in this boundary condition that, although $\partial R/\partial z$ is small, it multiplies a compensatingly large velocity, $u_z = O(u_r l/R)$. In the case of axisymmetric straining motion ($e = 0$) the disturbance flow outside the droplet is known to be that due to a line source on the axis, with a strength varying slowly along the length of the droplet (cf. Acrivos & Lo 1978). When the problem is not axisymmetric the two-dimensional disturbance flow is more complicated, involving, for example, line distributions of dipoles and stresslets in the $r, \theta$ plane.

The flow inside the slender droplet, which is quasi-unidirectional, is driven on the surface $r = R(\theta, z)$ by the undisturbed axial flow $u_z = Ez$, and opposed by an axial pressure gradient. In a long slender droplet, the integral of this pressure gradient produces a large pressure $p(z)$ which depends only on axial position and which dominates the deviatoric part of the stress tensor. Thus the stress boundary condition on the two-dimensional disturbance flow outside the droplet simplifies to

$$(\sigma_{rr} n_r + \sigma_{r\theta} n_\theta, \sigma_{r\theta} n_r + \sigma_{\theta\theta} n_\theta) = (\gamma K - p)(n_r, n_\theta) \quad \text{at} \quad r = R.$$

This equation is hereafter referred to as *the* stress boundary condition.

In the case of axisymmetric droplets, the interior axial flow is

$$u_z = -\frac{1}{\mu_i}\frac{dp}{dz}\frac{R^2 - r^2}{4} + Ez,$$

which produces a volume flux along the axis

$$q = -\frac{1}{\mu_i}\frac{dp}{dz}\frac{\pi}{8}R^4 + Ez\pi R^2.$$

Moreover, it is easy to show that, if the droplet is nearly axisymmetric, the expression for the volume flux differs from the above value only at second order in the deviations from symmetry, if $R$ is replaced by its average value over $\theta$, and this small change will be negligible in our calculations except in §4.3, where the correction will be given. Of course, in a steady droplet $q = 0$, and hence the internal pressure becomes

$$p = p_0 + 8\mu_i E \int_0^z \frac{z'}{R^2(z')}\,dz',$$

in which $p_0$ is a constant and for nearly axisymmetric droplets $R(z')$ is the value of $R(\theta, z')$ averaged over $\theta$.

The final condition to be applied to the solution of the governing equations is the volume normalization

$$\int_{-l}^{l}\int_0^{2\pi} \tfrac{1}{2}R^2\,d\theta\,dz = \frac{4\pi}{3}a^3.$$

Clearly, when the droplet is long and slender, $a$ is not the correct scale for lengths. We shall postpone the non-dimensionalization of the equations to the following sections, however, because different scalings are appropriate when the droplet is inviscid ($\mu_i = 0$) or slightly viscous ($\mu_i \ll \mu$).

## 3. An inviscid droplet

We first consider droplets with a very small interior viscosity, for which we may set $\mu_i = 0$. For such droplets the interior pressure is constant. We shall see in the following section that the droplet behaves in this inviscid way if $\mu_i \ll \mu(\gamma/E\mu a)^6$.

As indicated earlier, we shall treat the two-dimensional straining motion as a nearly axisymmetric flow by performing an asymptotic expansion in the parameter $e$, which will later be set equal to unity. In the following three subsections we recall the axisymmetric solution already obtained by Buckmaster (1972) and by Acrivos & Lo (1978) and then calculate the $O(e)$ and $O(e^2)$ corrections.

### 3.1. *The axisymmetric solution*

When the flow is axisymmetric ($e = 0$) and the droplet is also axisymmetric

$$R(\theta, z) = R_0(z)$$

the disturbance flow is that due to a line distribution of sources $Q(z)$ in $|z| \leqslant l$. The flow disturbance near to the surface of the droplet is thus a radial flow in the $r, \theta$ plane, $u_r(r, z) = Q(z)/2\pi r$, $u_z = 0$, with an associated stress disturbance

$$\sigma_{rr} = -\sigma_{\theta\theta} = -\mu Q/\pi r^2.$$

Adding this disturbance stress to the stress of the undisturbed flow and substituting into the stress boundary condition yields

$$-\mu E - \frac{\mu Q}{\pi R_0^2} = \frac{\gamma}{R_0} - p_0,$$

in which $p_0$ is the constant pressure inside the droplet. This equation is solved for $Q$, which can then be substituted into the expression for the flow disturbance. Now adding this flow disturbance to the undisturbed flow and substituting into the kinematic boundary condition yields

$$0 = -\tfrac{1}{2}ER_0 + \frac{R_0}{2\mu}(p_0 - \mu E) - \frac{\gamma}{2\mu} - Ez\frac{dR_0}{dz},$$

i.e.

$$Ez\frac{dR_0}{dz} - \left(\frac{p_0}{2\mu} - E\right)R_0 = -\frac{\gamma}{2\mu}.$$

The general solution to this equation satisfying the volume normalization is

$$R_0(z) = a\left(\frac{\gamma}{E\mu a}\right)\frac{1}{2\nu}\left(1 - \left|\frac{z}{l}\right|^{\nu}\right),$$

with

$$l = a\left(\frac{E\mu a}{\gamma}\right)^2 \tfrac{4}{3}(\nu+1)(2\nu+1),$$

where $\nu = p_0/2\mu E - 1$ is an arbitrary constant. Taylor (1964) chose $\nu = 2$ without giving any reason. The non-uniqueness was pointed out by Buckmaster (1972), and he suggested that it would be desirable for the shape to be analytic, i.e. for $\nu$ to be an even integer. Recently, Acrivos & Lo (1978) have shown that the above solution does not apply near $z = 0$ if $\nu$ is not an even integer and that another solution must be constructed in that region. Matching between these two solutions cannot be achieved, however, and hence only even integer values of $\nu$ are permissible, of which only the choice $\nu = 2$ leads to a stable solution. We therefore take the unique stable solution with $\nu = 2$ and perturb about that solution for $e \neq 0$. Note that for the droplet to be long and slender as assumed we require that the slenderness ratio $R_0(0)/l$ be small, or equivalently that $80\,(E\mu a/\gamma)^3 \gg 1$.

### 3.2. *The first perturbation*

We now pose an expansion in $e$ for $e$ small. Since in the undisturbed flow there is an $O(e)$ second-harmonic variation in $\theta$ which renders the solution invariant to the transformation $e = -e$ and $\theta = \theta + \frac{1}{2}\pi$, the change in shape must be of the form

$$R(\theta, z; e) = R_0(z)\,[1 + ef(z)\cos 2\theta + O(e^2)],$$

where, for convenience, we have factorized out the axisymmetric solution $R_0(z)$ found in the preceding subsection. This perturbed shape has a normal

$$(n_r, n_\theta) = (1, 2ef\sin 2\theta) + O(e^2)$$

in the $r, \theta$ plane and an average curvature $K = (1 + 3ef\cos 2\theta + O(e^2))/R_0$. Thus with $p_0 = 6\mu E$, corresponding to $\nu = 2$, and $T(z) = \gamma/\mu E R_0$, the right-hand side of the stress boundary condition becomes

$$\mu E\{(T-6, 0) + e(3fT\cos 2\theta, 2f(T-6)\sin 2\theta) + O(e^2)\},$$

in which changes in the internal pressure have been ruled out at $O(e)$ because the internal pressure cannot vary with $\theta$.

The general disturbance flow in the $r, \theta$ plane with a second-harmonic variation in $\theta$ is described by two line distributions of singularities $A(z)$ and $B(z)$ corresponding to stresslets and source quadrupoles:

$$u_r = EeR_0(A\rho + B\rho^3)\cos 2\theta,$$
$$u_\theta = EeR_0(\qquad B\rho^3)\sin 2\theta,$$

with associated stress fields

$$\sigma_{rr} = -2\mu Ee(2A\rho^2 + 3B\rho^4)\cos 2\theta,$$
$$\sigma_{r\theta} = -2\mu Ee(A\rho^2 + 3B\rho^4)\sin 2\theta,$$
$$\sigma_{\theta\theta} = 2\mu Ee(\qquad 3B\rho^4)\cos 2\theta,$$

in which $\rho = R_0/r$. We can now determine the left-hand side of the stress boundary condition to $O(e)$, including the undisturbed flow and the disturbed flow, and evaluating the unperturbed axisymmetric stress field on the $O(e)$ perturbed surface and the perturbation stress field on the unperturbed surface,

$$\mu E\{(T-6, 0) + e(\cos 2\theta[10f - 1 - 4A - 6B - 2fT], \quad \sin 2\theta[8f + 1 - 2A - 6B - 2fT]) + O(e^2)\}.$$

Equating the two sides of the stress boundary conditions yields

$$A = -5f - 1 - \tfrac{1}{2}fT, \quad B = 5f + \tfrac{1}{2} - \tfrac{1}{2}fT,$$

which can be substituted into the expressions for the perturbation flow. If we now evaluate the kinematic boundary condition to $O(e)$ we find an equation governing the shape function $f(z)$:

$$z\,df/dz + 5f = -1,$$

whose solution with $f$ finite at the centre of the droplet $z = 0$ is $f = -\frac{1}{5}$. Thus the cross-sectional shape does not change along the length of the droplet at $O(e)$.

### 3.3. *The second perturbation*

The $O(e)$ analysis above ignored quadratic terms $O(e^2)$. But since quadratics of second harmonics in $\theta$ produce zeroth and fourth harmonics in $\theta$, we are led to consider changes in shape at $O(e^2)$ of the form

$$R(\theta, z; e) = R_0(z)\,[1 - \tfrac{1}{5}e\cos 2\theta + e^2(g(z) + h(z)\cos 4\theta) + O(e^3)],$$

in which we have included the result of the $O(e)$ analysis that $f = -\frac{1}{5}$. For this shape the $O(e^2)$ change in the normal in the $r, \theta$ plane is $e^2(-\frac{1}{25} + \frac{1}{25}\cos 4\theta, (-\frac{1}{25} + 4h)\sin 4\theta)$ and the $O(e^2)$ change in the average curvature is $e^2(-\frac{1}{10} - g + (-\frac{9}{50} + 15h)\cos 4\theta)/R_0$. Thus the $O(e^2)$ change in the right-hand side of the stress boundary condition is

$$\mu E e^2\{T(-\tfrac{7}{50} - g) - p_2/\mu E + \tfrac{6}{25} + [T(-\tfrac{7}{50} + 15h) - \tfrac{6}{25}]\cos 4\theta, \quad (T(\tfrac{2}{25} + 4h) + \tfrac{6}{25} - 24h)\sin 4\theta\},$$

where $p_2$ is the $O(e^2)$ change in the pressure inside the droplet, which, of course, is constant.

The general disturbance flow in the $r, \theta$ plane with a zeroth- and fourth-harmonic variation in $\theta$ is described by three line distributions of singularities $F(z)$, $G(z)$ and $H(z)$:

$$u_r = Ee^2R_0\{F\rho + (G\rho^3 + H\rho^5)\cos 4\theta\},$$
$$u_\theta = Ee^2R_0(\quad \tfrac{1}{2}G\rho^3 \quad + H\rho^5)\sin 4\theta,$$

with associated stress fields

$$\sigma_{rr} = -2\mu Ee^2\{F\rho^2 + (\tfrac{9}{2}G\rho^4 + 5H\rho^6)\cos 4\theta\},$$
$$\sigma_{r\theta} = -2\mu Ee^2(\qquad 3G\rho^4 + 5H\rho^6)\sin 4\theta,$$
$$\sigma_{\theta\theta} = +2\mu Ee^2\{F\rho^2 + (\tfrac{3}{2}G\rho^4 + 5H\rho^6)\cos 4\theta\}.$$

Evaluating the stress on the perturbed surface, we find the $O(e^2)$ change in the left-hand side of the stress boundary condition to be

$$\mu Ee^2\{T(-\tfrac{7}{50} - 2g) + \tfrac{17}{50} + 10g - 2F + [T(-\tfrac{19}{50} - 2h) + \tfrac{73}{50} + 10h - 9G - 10H]\cos 4\theta,$$
$$[T(-\tfrac{7}{25} - 4h) + \tfrac{26}{25} + 16h - 6G - 10F]\sin 4\theta\}.$$

Equating the two sides of the stress boundary condition then yields

$$F = -\tfrac{1}{2}Tg + \tfrac{1}{20} + 5g + p_2/2\mu E,$$
$$G = T(\tfrac{1}{25} - 3h) + \tfrac{3}{10} - 10h,$$
$$H = T(-\tfrac{3}{50} + h) - \tfrac{1}{10} + 10h,$$

which can be substituted into the expressions for the perturbation flow. If we now evaluate the kinematic boundary condition to $O(e^2)$, we obtain the following equations governing the shape functions $g(z)$ and $h(z)$:

$$z\frac{dg}{dz} = \frac{p_2}{2\mu E} + \frac{\gamma}{\mu ER_0}(\tfrac{1}{100} + \tfrac{1}{2}g),$$

$$z\frac{dh}{dz} = \tfrac{3}{20} - 5h + \frac{\gamma}{\mu ER_0}(\tfrac{3}{100} - h).$$

The solution with $g$ and $h$ finite at $z = l$, the tip of the droplet, and analytic at its centre $z = 0$ is $g = -\frac{1}{50}$, $h = \frac{1}{100}$ and $p_2 = 0$. Thus the cross-sectional shape does not change along the length of the droplet even at $O(e^2)$. We have wondered, of course, whether the cross-sectional shape is constant at all orders in $e$, but an examination of the governing equations shows this to be unlikely.

The perturbation to the shape of the droplet changes the cross-sectional area at $O(e^2)$ from $\pi R_0^2(z)$ to $\pi R_0^2(z)(1 - \frac{1}{50}e^2)$. The $O(e)$ shape perturbation contributes an amount $\frac{1}{50}e^2$ to this relative change, while the remainder, $-\frac{2}{50}e^2$, is due to the $O(e^2)$ term of the expansion of $R$. This area change will lengthen the drop, so that

$$l = 20a(E\mu a/\gamma)^2(1 + \tfrac{1}{50}e^2).$$

Let us finally apply the small $e$ perturbation analysis to the case $e = 1$, which appears to be permissible since $e$ occurs as $\frac{1}{5}e$ in the result. The shape of the droplet in the two-dimensional straining motion is then predicted to be

$$R(\theta, z) = 0{\cdot}25a\left(\frac{\gamma}{E\mu a}\right)\left(1 - \left(\frac{z}{l}\right)^2\right)(0{\cdot}98 - 0{\cdot}2\cos 2\theta + 0{\cdot}03\cos 4\theta),$$

with

$$l = 20{\cdot}4a(E\mu a/\gamma)^2.$$

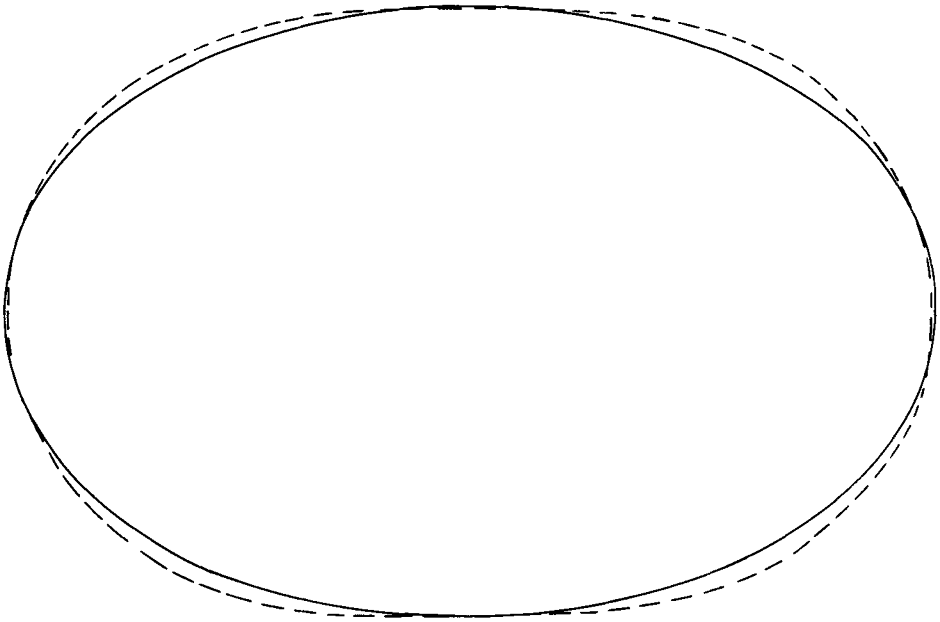

FIGURE 1. The cross-sectional shape of an inviscid droplet.
——, $O(e^2)$ prediction; ---, $O(e)$ prediction.

This cross-sectional shape is plotted in figure 1 as the full curve, together with the predictions of the less accurate $O(e)$ theory evaluated at $e = 1$, plotted as a dashed curve. It can be seen that the $O(e^2)$ terms offer very little improvement on the shape as determined by the $O(e)$ theory.

## 4. A slightly viscous droplet

Here we repeat the analysis of the preceding section, but now include the pressure variations along the length of the droplet which occur when the fluid in the droplet has a small viscosity. Again we first recall the axisymmetric solution and then in separate subsections calculate the $O(e)$ and $O(e^2)$ corrections.

### 4.1. *The axisymmetric solution*

We recall from §3.1 that an axisymmetric droplet $R(\theta, z) = R_0(z)$ disturbs the axisymmetric flow $e = 0$ in a way corresponding to a line distribution of sources. The strengths $Q(z)$ of the sources are calculated from the stress boundary condition, which must now include the pressure variations along the length of the droplet found at the end of §2. Thus

$$\frac{Q}{\pi R_0^2} = \frac{p_0}{\mu} + 8\lambda E \int_0^z \frac{z'\,dz'}{R_0^2(z')} - E - \frac{\gamma}{\mu R_0},$$

in which $\lambda = \mu_i/\mu \ll 1$. Substituting this calculated flow into the kinematic boundary condition yields an equation for $R_0(z)$, the shape of the droplet:

$$z\frac{dR_0}{dz} - \left(\frac{p_0}{2\mu E} - 1 + 4\lambda \int_0^z \frac{z'\,dz'}{R_0^2(z')}\right) R_0 = -\frac{\gamma}{2\mu E}.$$

As in the case of the inviscid droplets, there is a solution for each value of $p_0$, the pressure at the centre of the droplet. However, the shape is analytic at $z = 0$ only for an infinity of discrete values of $p_0$, and moreover only one of these values corresponds to a stable shape. This unique stable analytic shape, with the volume correctly normalized, is (Buckmaster 1973; Acrivos & Lo 1978)

$$R_0(z) = \tfrac{1}{2}a\lambda^{\frac{1}{6}}5^{\frac{1}{2}}\alpha^{-\frac{1}{2}}[1-\zeta^2] \quad \text{with} \quad \zeta = z\lambda^{\frac{1}{3}}/a\alpha,$$

in which the non-dimensional length $\alpha$ is related implicitly to the flow strength by

$$\frac{E\mu a}{\gamma}\lambda^{\frac{1}{6}} = \left(\frac{1}{20}\right)^{\frac{1}{2}}\frac{\alpha^{\frac{1}{2}}}{1+\frac{4}{3}\alpha^3}.$$

At low flow strengths $E\mu a/\gamma \ll \lambda^{-\frac{1}{6}}$, we recover the inviscid-droplet result

$$l = 20a(E\mu a/\gamma)^2.$$

As the flow strength increases, the droplet lengthens faster than is predicted by the inviscid analysis, because the larger pressure that now exists everywhere within the droplet (compared with the corresponding inviscid case) must be balanced by a further increase in the surface-tension force, i.e. by a further decrease in the local radius of the droplet and, therefore, by a further increase in its length. However, since, as can be easily verified, $p$ is linear in $\lambda l^3$, a further increase in the pressure is thereby induced. Thus a critical flow strength $E\mu a/\gamma = 0{\cdot}148\lambda^{-\frac{1}{6}}$ corresponding to $\alpha = 0{\cdot}630$ is attained beyond which a steady equilibrium shape cannot exist. This is also indicated by the fact that the implicit equation for $\alpha$ has no real roots for

$$E\mu a/\gamma > 0{\cdot}148\lambda^{-\frac{1}{6}}.$$

Droplets with a non-dimensional length $\alpha$ exceeding $0{\cdot}630$ correspond to equilibria for subcritical flow strengths, but these equilibria are unstable to length changes, droplets longer than the equilibrium value extending indefinitely and those with shorter lengths contracting to the equilibrium value at the same flow strength with $\alpha < 0{\cdot}630$ (Acrivos & Lo 1978). The breakup criterion obtained above applies, of course, only if the corresponding slenderness ratio $R_0(0)/l$ is small, which, in view of our solution, requires that $(5\lambda)^{\frac{1}{2}} \ll 1$.

### 4.2. *The first perturbation*

As in §3.2, we pose an expansion in $e$ for $e$ small, with the shape again given by

$$R(\theta, z; e) = R_0(z)\,[1+ef(z)\cos 2\theta + O(e^2)].$$

The formulae in §3.1 for the normal and curvature in terms of $f(z)$ remain unchanged. Thus the right-hand side of the stress boundary condition becomes

$$\mu E\{(T-P, 0) + e[3fT\cos 2\theta, 2f(T-P)\sin 2\theta] + O(e^2)\},$$

in which $T(z) = \gamma/\mu E R_0(z)$ represents, as before, the contribution from the surface-tension forces and $P(z) = 6+\frac{16}{5}\alpha^3/(1-\zeta^2)$ is the internal pressure, divided by $\mu E$, which now varies along the droplet.

The general flow disturbance in the $r, \theta$ plane can be again represented by line

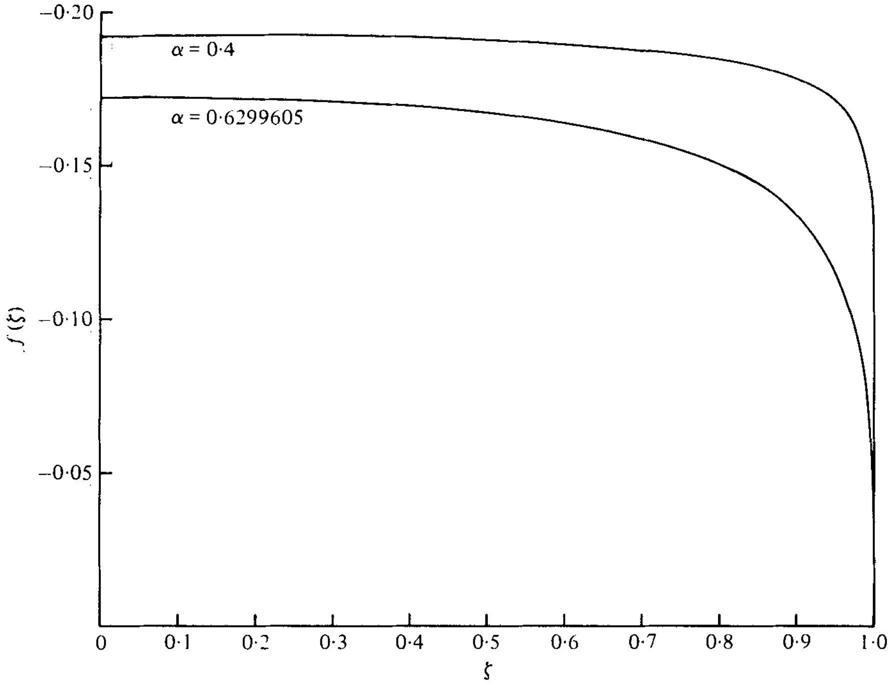


FIGURE 2. The shape function $f(\zeta)$ for various values of $\alpha$.

distributions $A(z)$ and $B(z)$ of stresslets and source quadrupoles as in § 3.2. Thus the left-hand side of the stress boundary condition becomes

$$\mu E\{(T-P, 0)+e(\cos 2\theta[2fP-2f-1-4A-6B-2fT], \\ \sin 2\theta[2fP-4f+1-2A-6B-2fT])+O(e^2)\}.$$

Equating the two sides of the stress boundary condition yields

$$A = -fP+f-1-\tfrac{1}{2}fT, \quad \text{and} \quad B = fP-f+\tfrac{1}{2}-\tfrac{1}{2}fT,$$

so the perturbation flow is determined in terms of $f$. Finally, by evaluating the kinematic boundary condition to $O(e)$, we obtain the equation governing the shape function $f(z)$:

$$z\,df/dz+(P-1)f = -1,$$

whose solution with $f$ finite at $z = 0$ is

$$f = -\frac{(1-\zeta^2)^{\frac{8}{5}\alpha^3}}{\zeta^{5+\frac{16}{5}\alpha^3}}\int_0^\zeta \frac{t^{4+\frac{16}{5}\alpha^3}}{(1-t)^{\frac{8}{5}\alpha^3}}\,dt.$$

Thus, for a slightly viscous droplet the cross-sectional shape varies along the length of the droplet, from a maximum perturbation of $f = -1/(5+\frac{16}{5}\alpha^3)$ at the centre to a zero perturbation $f = 0$ at the ends (if $\alpha \neq 0$). However, in the range of interest, $0 < \alpha < 0{\cdot}630$, $f$ deviates little from its larger inviscid value of $-\frac{1}{5}$, except within a small region $(\frac{1}{2})^{1+5/8\alpha^3}$ from the end, where $f$ drops to zero (cf. figure 2).

### 4.3. *The second perturbation. Comparison with experimental results*

Following §3.3, we consider a cross-sectional shape to $O(e^2)$ given by

$$R(\theta, z; e) = R_0(z)[1+ef(z)\cos 2\theta + e^2(g(z)+h(z)\cos 4\theta)+O(e^3)],$$

in which $f(z)$ is the function found above. For this shape, the $O(e^2)$ change in the unit normal is $e^2(-f^2+f^2\cos 4\theta, (-f^2+4h)\sin 4\theta)$, and that in the average curvature $e^2(-\frac{5}{2}f^2-g+(-\frac{9}{2}f^2+15h)\cos 4\theta)/R_0$. The $O(e)$ change in shape with zero mean in $\theta$ and the $O(e^2)$ mean change in shape both alter the internal pressure gradient at $O(e^2)$, thus the internal pressure changes by

$$\mu E e^2 P_2 = \mu E e^2\left[p_2/\mu E + \tfrac{32}{5}\alpha^3\int_0^\zeta \zeta'(\tfrac{3}{2}f^2-2g)/(1-\zeta'^2)^2\,d\zeta'\right],$$

which includes a change $e^2 p_2$ in the constant pressure. Combining these effects leads to an $O(e^2)$ change in the right-hand side of the stress boundary condition:

$$\mu E e^2\{T(-\tfrac{7}{2}f^2-g)+Pf^2-P_2+[T(-\tfrac{7}{2}f^2+15h)-Pf^2]\cos 4\theta,$$
$$[T(2f^2+4h)+P(f^2-4h)]\sin 4\theta\}.$$

The general flow disturbance in the $r, \theta$ plane can again be represented by line distributions of singularities $F(z)$, $G(z)$ and $H(z)$ as in §3.3. Thus the left-hand side of the stress boundary condition becomes

$$\mu E e^2\{T(-\tfrac{7}{2}f^2-2g)+P(\tfrac{7}{2}f^2+2g)-\tfrac{5}{2}f^2+2f-2g-2F$$
$$+[T(-\tfrac{19}{2}f^2-2h)+P(\tfrac{19}{2}f^2+2h)-\tfrac{21}{2}f^2+2f-2h-9G-10H]\cos 4\theta,$$
$$[T(-7f^2-4h)+P(13f^2+4h)-12f^2+8f-8h-6G-10H]\sin 4\theta\}.$$

Equating the two sides of the stress boundary condition yields

$$F = -\tfrac{1}{2}Tg+P(\tfrac{5}{4}f^2+g)-\tfrac{5}{4}f^2+f-g+\tfrac{1}{2}P_2,$$
$$G = T(f^2-3h)+P(-\tfrac{1}{2}f^2-2h)+\tfrac{1}{2}f^2-2f+2h,$$
$$H = T(-\tfrac{3}{2}f^2+h)+P(\tfrac{3}{2}f^2+2h)-\tfrac{3}{2}f^2+2f-2h,$$

so the $O(e^2)$ perturbation flow is determined in terms of $g$ and $h$. Finally, evaluating the kinematic boundary condition to $O(e^2)$ produces equations governing the shape functions $g(z)$ and $h(z)$:

$$z\,dg/dz = T(\tfrac{1}{4}f^2+\tfrac{1}{2}g)+\tfrac{3}{2}Pf^2-\tfrac{3}{2}f^2+\tfrac{3}{2}f+\tfrac{1}{2}P_2,$$
$$z\,dh/dz = T(\tfrac{3}{4}f^2-h)+P(-\tfrac{3}{4}f^2-h)+\tfrac{3}{4}f^2-\tfrac{3}{2}f+h.$$

It can be seen by direct substitution that $h = \frac{3}{4}f^2$; however, the function $g(z)$ must be determined numerically and is shown in figure 3 for $\alpha = 0{\cdot}630$, the critical value of $\alpha$ in the axisymmetric case.

The perturbation to the shape of the droplet changes the cross-sectional area at $O(e^2)$. If we apply the volume normalization and keep the length as $a\alpha\lambda^{-\frac{1}{3}}$ ($\alpha$ will of course change slightly), we are forced to modify $R_0$ slightly to

$$R_0(z) = \tfrac{1}{2}a\lambda^{\frac{1}{6}}5^{\frac{1}{2}}\alpha^{-\frac{1}{2}}(1-\tfrac{1}{2}e^2q(\alpha))(1-\zeta^2),$$

where $\quad q(\alpha) = \frac{15}{8}\int_0^1 (1-\zeta^2)^2(2g+\frac{1}{2}f^2)\,d\zeta \quad$ with $\quad q(0{\cdot}630) = -0{\cdot}0557.$

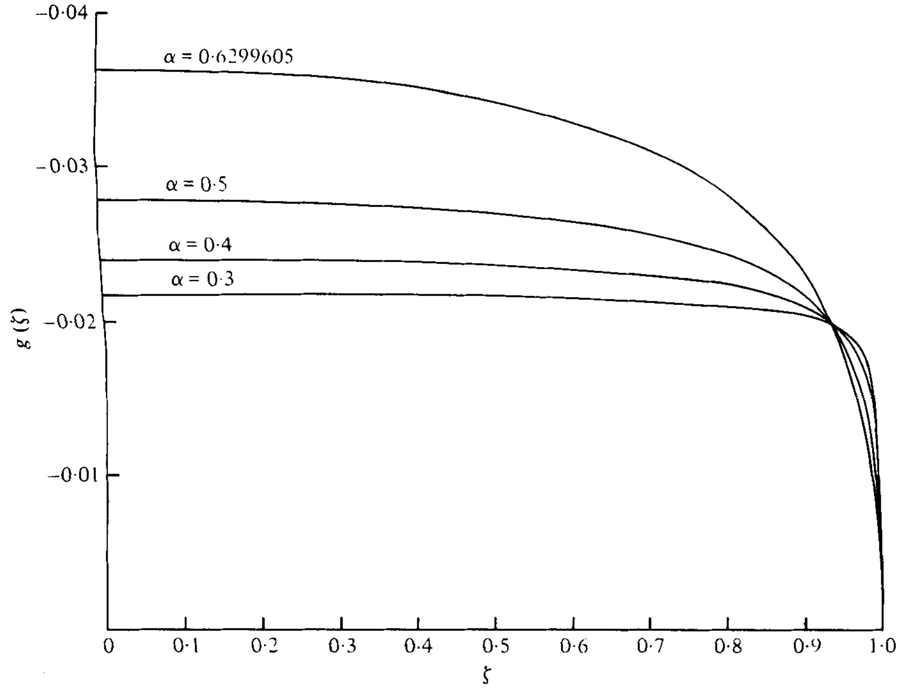


FIGURE 3. The shape function $g(\zeta)$ for various values of $\alpha$.

The implicit relation between $\alpha$ and the flow strength becomes then

$$(\tfrac{1}{20})^{\frac{1}{2}}\,\frac{\gamma}{E\mu a}\,\lambda^{-\frac{1}{6}} = \alpha^{-\frac{1}{2}} + \tfrac{4}{5}\alpha^{\frac{5}{2}} + \tfrac{1}{2}q\,e^{2}(\tfrac{4}{5}\alpha^{\frac{5}{2}} - \alpha^{-\frac{1}{2}}).$$

At $e = 1$, this gives a critical flow strength

$$E\mu a/\gamma = 0{\cdot}145\lambda^{-\frac{1}{6}} \quad \text{at} \quad \alpha = 0{\cdot}635,$$

which is essentially the same expression as that found earlier by Taylor (1964) and by Acrivos & Lo (1978) for the axisymmetric case. This was to be expected since, as we have already seen, the constants of proportionality between $l/a$ and $(E\mu a/\gamma)^2$ for an inviscid droplet in, respectively, an extensional and a hyperbolic flow differ by only 2%.

The only experimental data on critical flow strengths at low values of $\lambda$ with which our theoretical results can be compared are those reported by Grace (1971) and by Yu (1974). These are reproduced in figure 4, where it is apparent that, for $\lambda \leqslant 1$, they can be accurately represented by

$$E\mu a/\gamma = 0{\cdot}12\lambda^{-0{\cdot}16}.$$

This is almost exactly the theoretical expression except that the respective proportionality constants differ by about 18%. There are, of course, a number of possible reasons for this relatively small discrepancy. To begin with, our theoretical analysis leads to only a *sufficient* criterion for breakup since the possibility of a steady droplet shape being unstable to finite amplitude disturbances has not been excluded. In addition, though, the experimental results may be susceptible to error owing to the

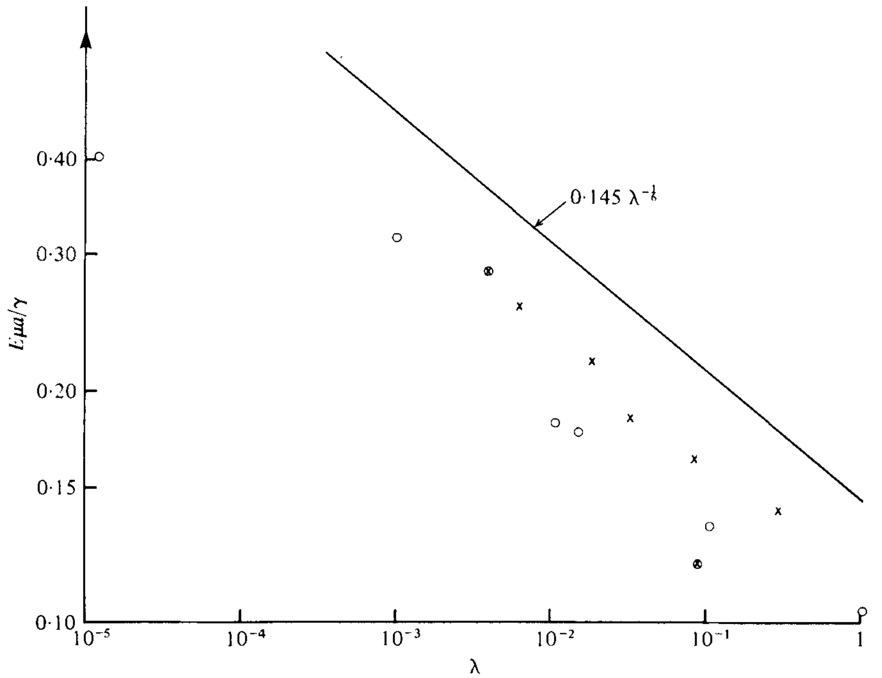


FIGURE 4. Comparison between theoretical prediction and the experiments of Grace (1971) (⊗, $\mu$ = 502·5 P; ○, $\mu$ = 45·5 P) and Yu (1974) (×, $\mu$ = 103 P).

presence of unavoidable small surface-active impurities, which invariably tend to lower the surface tension $\gamma$ during the course of an experiment. Thus the reported values of $E\mu a/\gamma$ at breakup could be too low if higher values of $\gamma$, pertaining to cleaner systems, were used in calculating this dimensionless group. At any rate, considering the limitations of our theory and the difficulties involved in the experimental measurements, the agreement between theory and experiment should be viewed as gratifying.

We close with a few remarks about inertia effects. As shown by Acrivos & Lo (1978), the case of axisymmetric flow past an inviscid slender droplet of zero density can also be treated analytically by their technique when the external Reynolds number is finite, by simply adding to the normal-stress balance the pressure of undisturbed flow, $-\frac{1}{2}\rho E^2 z^2$, where $\rho$ is the density of the external fluid. The equation for $R_0$ then becomes

$$z\frac{dR_0}{dz} - \left(\nu + \frac{\rho E}{4\mu} z^2\right) R_0 = -\frac{\gamma}{2\mu E},$$

where $\nu = p_0/2\mu E - 1$ is, as before, a constant parameter. The solution of the above is

$$R_0(\zeta) = \frac{\gamma}{2\mu E}\, \zeta^\nu \exp\left(\tfrac{1}{8}\beta\zeta^2\right) \int_\zeta^1 t^{-\nu-1} \exp\left(-\tfrac{1}{8}\beta t^2\right) dt,$$

where $\zeta = z/l$ and $\beta = \rho E l^2/\mu$, plus the constant volume condition

$$\frac{l}{a} = \frac{2}{3}\left(\frac{E\mu a}{\gamma}\right)^2 \Bigg/ \int_0^1 \left(\frac{\mu E}{\gamma} R_0(\zeta)\right)^2 d\zeta.$$

The requirement that $R_0$ be analytic at the origin determines $\nu$, which is found to vary from 2 (in the inviscid limit) to approximately 2·52 at the point of breakup $\beta = 1{\cdot}6$, or $(E\mu a/\gamma)(\rho a\gamma/\mu^2)^{\frac{1}{6}} = 0{\cdot}284$ (cf. Acrivos & Lo 1978).

For the non-axisymmetric case the solution can again be expanded in $e$. Proceeding as before, we find that

$$f = -\zeta^{-(2\nu+1)}\exp\left(-\tfrac{1}{4}\beta\zeta^2\right)\int_0^\zeta t^{2\nu}\exp\left(\tfrac{1}{4}\beta t^2\right)dt$$

and that $h = \frac{3}{4}f^2$; also, the equation for $g$ remains unchanged except that $P_2$ is now a constant. However, since in the range of interest ($2 \leqslant \nu \leqslant 2{\cdot}52$ and $0 \leqslant \beta \leqslant 1{\cdot}6$) $|f| \leqslant \frac{1}{5}$, we should expect the breakup criterion to be essentially identical with that found earlier in the corresponding axisymmetric case.

Thus, although the local cross-section of the droplet does become significantly non-circular when the impressed flow is altered from extensional to hyperbolic, the theoretical criterion for breakup remains effectively unchanged in all respects provided, of course, that the droplet is slender.

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