S shape of a granular pile in a rotating drum

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The shape of a granular pile in a rotating drum is investigated. Using discrete elements method (DEM) simulations we show that the "S shape" obtained for high rotation speed can be accounted for by the friction on the end plates. A theoretical model which accounts for the effect of the end plates is presented and the equation of the shape of the free surface is derived. The model reveals a dimensionless number which quantifies the influence of the end plates on the shape of the pile. Finally, the scaling laws of the system are discussed and numerical results support our conclusions.

Among all the geometries used to study granular flows, the rotating drum might be the most complex [1–3]. Depending on the angular velocities, two different regimes occur. At low rotation speed the free surface of the pile is inclined and flat, but a significant curvature appears at high rotation speed: the so-called S shape [4–6]. Up until now this transition remains relatively unexplored [1,7]. Different explanations have been proposed, but no consensus has been reached. One possible origin of the curvature of the bed is the centrifugal forces acting on the granular flow. This effect can be quantified by the Froude number: $FR = R\Omega^2/g$, where Ω is the angular rotation speed of the drum, R its radius, and g the gravitational acceleration. Another possible cause of the S shape is the "feeding inertia" of the grains. If the rotation speed is high then the velocity of the grains, v, can be very high and the grains can display ballistic trajectories. The vertical distance, Δ, traveled by such a grain should be compared to the radius of the drum. This effect is again quantified by the Froude number: $\Delta/R = v^2/(2gd) = R^2\Omega^2/(2gR) = FR/2$. Although these effects can indeed account for the curvature of the free surface, the S shape can be observed for very low values of the Froude number (typically, $FR = 10^{-4}$, see [7], or $FR = 10^{-3}$ in Fig. 2). This indicates that, in this case, the centrifugal force is not the origin of the S shape. Another explanation is then needed.

Several other models have been proposed [2,8–10] and were able to recover the S shape. Yet, none has taken into account the influence of the end plates. Here instead, we show that in short drums, the S shape can be explained by the friction of the end plates of the cylinders. This idea is based on recent work on confined granular flows which have shown that the sidewalls of a channel can drastically influence the flow properties [11–16]. In particular, Khakhar *et al.* [11] reported that the inclination of a flow on a heap increases with increasing flow rate, a phenomenon later explained by Taberlet *et al.* [13]. These authors have derived a law linking the inclination of the free surface, φ, with the flow thickness, h, and the channel width, L,

$$\tan \varphi = \mu_i + \mu_w \frac{h}{L}, \tag{1}$$

where μ_i and μ_w are two effective friction coefficients describing the internal and sidewalls frictional properties, respectively (see [13] for details). In a rotating drum, the flow rate is the highest near the center of the drum, which is also the point at which the free surface is the steepest. This supports the idea that the end plates can have a crucial influence on the shape of the free surface. The outline of the paper is as follows: first the simulation method is presented. The crucial effect of the end plates on the shape of the pile is then evidenced by two numerical tests: one with frictional and one with frictionless end plates. A theoretical model is presented and the equation of the free surface is derived. Finally, the scaling laws of the system are discussed.

In this paper, we consider a cylindrical drum of radius R which rotates at a constant angular velocity Ω. The drum is partially filled with granular material. The position along the horizontal and vertical axes are labeled x and y, respectively, and the local slope of the free surface is denoted $\varphi(x)$ (see Fig. 1). For all the results shown in this paper, Ω is large enough to produce continuous flows but small enough to neglect centrifugal forces and the feeding inertia effects, the Froude number being kept below 1.

The shape of the free surface of the pile is investigated through numerical simulations of soft-sphere molecular dynamics method. Although not flawless, this type of simulation has been widely used in the past two decades and has proven to be very reliable for the study of granular flows in a rotating drum [17–19]. The forces acting between two colliding grains are computed from the normal overlap, δ_n, and the equations of motion (displacement and rotation) are integrated using the Verlet method [20]. The schemes used for the forces calculations are the spring-dashpot and the regularized Coulomb laws, with the following values: particle
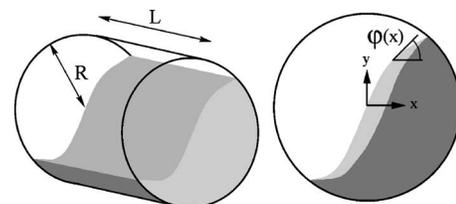
FIG. 1. We consider a half-filled three-dimensional (3D) drum of radius R and length L. The local slope of the free surface is denoted $\varphi(x)$. Right: light gray corresponds to flowing material and dark gray to solid rotation.
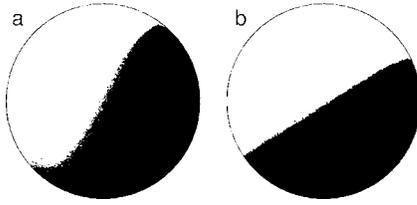
FIG. 2. Side views of the 3D drum. In both cases, $R=80d$, $L=10d$, $\mu=0.3$, $\Omega=0.1$ rad/s, and $N=36\,000$ grains. (a) frictional and (b) frictionless end plates. The difference in shape is very clear and originates from a change in the frictional properties of the end plates. This provides evidence of the crucial role of the end plates on the shape of the free surface.

diameter $d=8$ mm, mass$=0.16$ g, spring constant k_n $=40\,000$ N m^{-1}, viscous damping $\gamma_n=0.5$ s^{-1}, leading to a normal coefficient of restitution $e_n=0.64$, regularization constant $\gamma_t=5$ s^{-1}, time step $dt=10^{-6}$ s and, unless otherwise mentioned, $\mu=0.3$. Note that the restitution, i.e., the inelasticity, does not seem to play a role. Different values of e_n were tried ($0.3 < e_n < 0.8$) and did not affect the shape of the pile. The collisions against the wall and the end plates are treated like particle-particle collisions with one of the particles having infinite mass and radius, which mimics a flat surface. Note that the mechanical properties of the grain/end-plate collisions can be chosen independently of those of the grain/grain collisions. In particular, it is possible to simulate frictional grains placed in a drum with frictionless end plates.

The radius of the drum R is typically $100d$, and its length, L, is varied from $10d$ to $200d$. The grains are first released in the drum and rotation is started (at rotation speed Ω) only after they have settled. The number of grains is chosen so that the drum is half filled. Our simulations contain a large number of particles, between 5000 and 70 000, and run for typically five full rotations of the drum. The granular material is made slightly polydisperse with an equal number of grains of diameter d and $4/5d$, in order to avoid crystallization. The snapshots of the pile were taken after two full rotations after the shape has reached a steady state. On such short times no radial (or axial) segregation was observed. Note that no interstitial fluid is present in our simulation but its effect is expected to be negligible for the low values of Ω (typically FR$=10^{-3}$).

In order to demonstrate the crucial influence of the end plates, and in particular their frictional properties, two simulations differing only by their end-plate friction coefficients were performed. Figure 2 shows two runs with identical values of all parameters, one with frictional end plates (a) and the other with frictionless end plates (b). The two shapes are very different, displaying on one hand the S shape (a), and on the other hand a flat surface (b). The rotation speed being the same in both cases (meaning that FR is the same) this result gives strong evidence that the end plates have a crucial influence on the shape of the free surface and shows that neither the centifugal force nor the feeding inertia is responsible for the S shape.

Let us now present a theoretical model based on Eq. (1). The aim of our model is to derive the equation for the position of the free surface, that is, to find an expression for

$y_{surf}(x)$. The following notations will be used: $\bar{\rho}$ is the average density of the material (in kg m^{-3}), $q_{feed}(x)$ is the local feeding rate per unit length and width along the flow (in kg s^{-1} m^{-2}), $Q_{flow}(x)$ is the local flow rate per unit width (in kg s^{-1} m^{-1}), $\dot{\gamma}$ is the shear rate, $h(x)$ is the local thickness of the flow, g is the gravity, and d is the grain diameter.

Some authors mentioned the dependence of $\dot{\gamma}$ on the flow properties [7,21]. Yet, in many cases this dependence is weak [22] so, in order to simplify our analysis we assume that the shear rate is a constant. For geometrical reasons, the feeding rate, q_{feed}, increases linearly with the distance from the center of the drum, r, and is positive in one half of the drum ($x>0$) and negative in the other half. The feeding rate then reads

$$q_{feed}(x) = \bar{\rho}\Omega r. \tag{2}$$

Because of mass conservation, the flow rate at a given point of the free surface (at the position $\{x, y_{surf}(x)\}$) has to be equal to the integral of the feeding rate coming from the solid rotation between the outer cylinder and the considered radius, $r=\sqrt{x^2+y_{surf}^2}$,

$$Q_{flow}(x) = \int_r^R q_{feed}(r)dr = \frac{\bar{\rho}\Omega}{2}(R^2-r^2) = \frac{\bar{\rho}\Omega}{2}(R^2-x^2-y_{surf}^2). \tag{3}$$

To go further, one needs a relation between the flow thickness and flow rate. Previous theories for the shape, e.g., [8,9], have made different assumptions about the velocity profiles and the thickness of the flowing layer but those predate experimental observations. Here we build a model based on the observations in [3] which report a linear velocity profile with a near universal shear rate. Since we consider that the shear rate is a constant, dimensional analysis yields $\dot{\gamma} = a\sqrt{g/d}$. For simplicity we use $a=1$ but this value has no effect on the qualitative results presented below. Hence the mass flow in the layer of thickness h with the linear velocity gradient $\dot{\gamma}$ is

$$Q_{flow}(x) = \frac{1}{2}\bar{\rho}\dot{\gamma}h(x)^2 = \frac{\bar{\rho}}{2}\sqrt{\frac{g}{d}}h(x)^2. \tag{4}$$

The slope of the free surface (i.e., $\partial y_{surf}/\partial x$) is given by Eq. (1),

$$\frac{\partial y_{surf}}{\partial x} = \tan\varphi(x) = \mu_i + \mu_w\frac{h(x)}{L}. \tag{5}$$

Using (3) and (4), one can express h as a function of x and plugging it into (5) leads to

$$\frac{\partial y_{surf}}{\partial x} = \mu_i + \mu_w\left(\frac{d\Omega^2}{g}\right)^{1/4}\frac{1}{L}\sqrt{R^2-x^2-y_{surf}^2}. \tag{6}$$

Equation (6) shows that the slope has a minimum value μ_i at the outer cylinder and a maximum at the center. It can be numerically integrated and the shape of the pile can be plotted for different values of the parameters (see Fig. 3). The equation is integrated outwards starting from the center of the drum ($x=y=0$) using the Euler method. Note that no
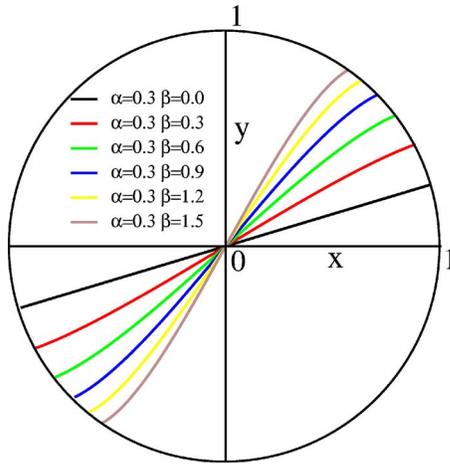
FIG. 3. (Color online) Numerical solutions for y_{surf} for $R=1$, $\mu_i=\alpha=0.3$, and various values of $\beta=\mu_w/L(d\Omega^2/g)^{1/4}$. For $\beta=0$, the influence of the end plates is null and the free surface is flat. When β is increased, i.e., increasing influence of the end plates, a curvature appears and the free surface displays an S shape.

additional condition is needed since the size of the drum is embedded in Eq. (6) while its symmetry ensures a total mass of 1/2. Note, however, that Eqs. (1)–(6) do not require any assumptions regarding the filling ratio. Therefore, Eq. (6) can be integrated from any point, for instance on the outer boundary, after what the corresponding filling ratio could be computed. For frictionless end plates (i.e., $\mu_w=0$), the free surface is a flat plane of equation $y=\mu_i x$, which confirms the results of Fig. 2. With increasing influence of the end plates a curvature appears and the pile displays an S shape, which qualitatively reproduces the experimental and numerical behavior.

One major criticism can be made regarding the present model: Eq. (1) was derived for uniform flows and its validity for flows in a rotating drum is questionable since the flow rate varies along the flow. Moreover, when the grains have acquired a high velocity during the flow, they can form an upward tail at the end of the slope. Therefore our model cannot reproduce this upward tail since the inertia of the grains is neglected.

The model indicates that there are different ways of changing the influence of the end plates. One way is to reduce their frictional properties as demonstrated by Fig. 2, and another way is to increase the length of the drum. Indeed, when L increases, the second term in Eq. (6) vanishes and the free surface should tend toward a flat plane. This is demonstrated by Fig. 4, which shows three runs with increasing drum length while keeping the radius, filling ratio, frictional properties, and rotation speed constant. In a narrow drum [Fig. 4(a), $L=10d$] the pile displays a well-marked S shape. In a longer drum [Fig. 4(b), $L=50d$] the free surface flattens although a curvature is still visible. When the length is further increased [Fig. 4(c), $L=300d$] the system relaxes to its ground state consisting of a flat surface. This fact gives further support to our model. Once again, since the radius and the rotation speed are identical for all three runs, FR is identical as well, which shows that in this case, the S shape originates neither from the centrifugal force nor the feeding inertia.
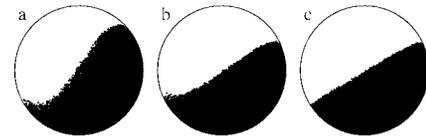
FIG. 4. Side views of the 3D drum. In all cases, $R=40d$, $\mu=0.3$, and $\Omega=0.2$ rad/s. (a) $L=10d$, (b) $L=50d$, (c) $L=300d$. The longer the channel, the flatter the surface. This result gives further support to the idea that the end plates are responsible for the S shape of the free surface.

Let us now discuss the scaling laws of the system. First, note that Eq. (6) can be made dimensionless using the reduced variables $\tilde{x}=x/R$ and $\tilde{y}=y_{surf}/R$,

$$
\begin{cases}
\dfrac{\partial \tilde{y}}{\partial \tilde{x}} = \mu_i + \mu_w \Lambda \sqrt{1 - \tilde{x}^2 - \tilde{y}^2} \\[2mm]
\Lambda = \left(\dfrac{d\Omega^2}{g}\right)^{1/4} \dfrac{R}{L}
\end{cases}
\tag{7}
$$

Equation (7) states that the free surface has a ground state consisting of a plane of slope μ_i from which it deviates when the end plates play an important role. Moreover, the shape of the free surface in dimensionless units depends only on the value of Λ which therefore contains all the scaling laws of the system. Among them, if the radius R is varied while keeping d, g, and L constant, the rotation speed should scale as the inverse of R^2: $\Omega \propto 1/R^2$. Similarly, one can find the following scaling laws linking the rotation speed to the drum length and particle diameter: $\Omega \propto L^2$ and $\Omega \propto 1/\sqrt{d}$. In particular, if the radius of the drum is doubled, the rotation speed should be reduced by a factor 4 in order to keep Λ constant and obtain identical shapes. Similarly, if the length
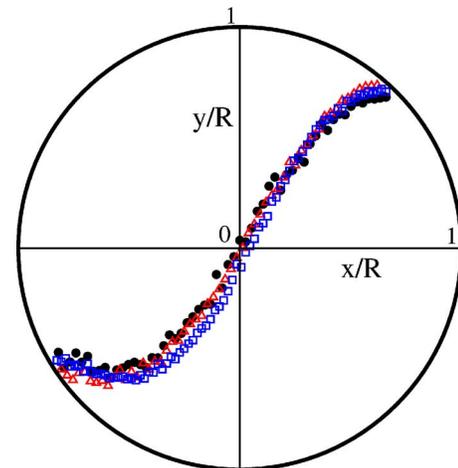
FIG. 5. (Color online) Rescaled plots of the position of the free surface for various values of R, L, and Ω: Triangles: $R=80d$, $L=10d$, $\Omega=0.05$ rad/s; squares: $R=80d$, $L=20d$, $\Omega=0.2$ rad/s; circles: $R=40d$, $L=10d$, $\Omega=0.2$ rad/s. In the simulation, the position of the free surface is calculated by identifying the highest grain located at a position x. R, L, and Ω are varied while keeping Λ constant. The data collapse onto one unique curve which validates the proposed scaling laws.

of the drum is doubled, the rotation speed has to increase by a factor of 4. These two examples are illustrated in Fig. 5, which shows rescaled plots of the free surface for various values of R, L, and Ω while keeping Λ constant. The different sets of data collapse onto one unique curve, giving numerical confirmation of the scaling laws inferred from Eq. (7). Let us mention here that the scaling law $\Omega \propto 1/R^2$ and $\Omega \propto 1/\sqrt{d}$ are compatible with experimental observations reported by Felix [7]. One can also notice that $\Lambda^4 = \mathrm{FR}d/R(R/L)^4$. With the Froude number and the ratio d/R being small, it appears that the end plates have a significant effect only if the aspect ratio R/L is large.

We have shown that frictional end plates have a major and nontrivial influence on the shape of a granular pile in a rotating drum. Through numerical simulations we have demonstrated that the S shape disappears when the friction on the end plates vanishes or when the drum is long enough. Our theoretical model supports the idea that the end plates are responsible for the curvature of the free surface and shows that the dimensionless number Λ (which includes all the relevant parameters: particle size, drum length and radius, rotation speed, and gravity) entirely describes the shape of the pile. Our theoretical analysis could be improved by including the inertia of the grains. A model similar to that of Khakhar *et al.* [23] could be adapted by adding a friction term accounting for the effect of the end plates. Such a model will be presented in a future paper. Finally, we have presented only a few scaling laws regarding the shape of the free surface but many more can be inferred from the expression of Λ.

ACKNOWLEDGMENTS

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