A coating flow on a rotating vertical disk

John Hinch & Matthew Crowe

CMS-DAMTP, University of Cambridge

November 13, 2014

Andy's speculation is correct, of course.

Coating flow flow on a rotating vertical disk

Lubrication theory for thin film $h(\mathbf{x}, t)$:

$$\frac{\partial h}{\partial t} + \nabla \cdot \left(\mathbf{\Omega} \times \mathbf{x} h + \tfrac{1}{3} \gamma h^3 \mathbf{g} \right) = -\tfrac{1}{3} \alpha \nabla \cdot (h^3 \nabla \nabla^2 h),$$

$$\int_{r \leq 1} h \, dA = \pi,$$

with small surface tension $\alpha = \sigma h_0^3 / \mu \Omega R^4 \ll 1$,
and gravity $\gamma = \rho g h_0^2 / \mu \Omega R^2$.

What is the maximum load, $\gamma_{\max}$, without dripping?

Previously

Parmar, Tirumkudulu & Hinch (2009) PoF 21, 103102.

Experiments: $\gamma_{\max} \approx 0.30$.
Numerics to $\gamma = 0.12$ with $\alpha = 10^{-5}$.
Asymptoics small γ, small α to $O(\gamma^3, \alpha\gamma^3)$ works to $\gamma = 0.06$.

Acrivos (2010) PoF Comment 22, 059101.

$$\gamma_{\max} = \left(\frac{8}{15}\right)^2 = 0.2844.$$

Zero surface tension, $\alpha = 0$

Steady if

$$\left(\boldsymbol{\Omega} \times \mathbf{x} h + \gamma h^3 \mathbf{g}\right) \cdot \nabla h = 0,$$

i.e. $h = \mathrm{const}$ on off-set circle,

$$\left|\mathbf{x} - \gamma h^2 \boldsymbol{\Omega} \times \mathbf{g}\right| = r,$$

circles of radius r, centred at $\mathbf{x} = (-\gamma h^2, 0)$.

How does h vary across circles, i.e. $h = H(r)$?

Acrivos: at maximum load, all circles must pass through $\mathbf{x} = (-1, 0)$,

$$\text{so} \quad H(r) = \frac{1}{\sqrt{\gamma_{\max}}} \sqrt{1 - r}$$

volume normalisation gives $\gamma_{\max} = (8/15)^2$.

Small surface tension, $\alpha \ll 1$

Study accumulative effect over a long time of small surface tension

$$h(\mathbf{x}, t) \sim H(r, T) + \alpha h_1(\mathbf{x}, T) \quad \text{with slow time} \quad T = \alpha t.$$

Governed by

$$\frac{\partial H}{\partial t} + \nabla \cdot \left(\mathbf{\Omega} \times \mathbf{x} h_1 + \tfrac{1}{3}\gamma h^3 \mathbf{g} \right) = -\tfrac{1}{3}\nabla \cdot (H^3 \nabla \nabla^2 H).$$

Integrate around off-set circle to eliminate second term on LHS
(GKB 56, LGL EJH 70)

Technical details

The off-set circles give non-orthogonal co-ordinates

$$\mathbf{x} = (r\cos\theta + X(r), r\sin\theta).$$

Metrics   sorry to repeat symbols h and α

$$h_r = \sqrt{1 + 2X'\cos\theta + X'^2}, \quad h_\theta = r, \quad h_r h_\theta \sin\alpha = r(1{+}X'\cos\theta).$$

Hence curvature $\kappa = \nabla^2 h$ for $h = H(r)$

$$= \frac{1}{r(1 + X'\cos\theta)}\left[\frac{\partial}{\partial r}\left(\frac{r}{1 + X'\cos\theta}\frac{dH}{dr}\right) + \frac{(X' + \cos\theta)X'}{(1 + X'\cos\theta)^2}\frac{dH}{dr}\right]$$

and this is only half potential terms.

And $X(r) = -\gamma H^2(r)$ with unknown H.

...technical details

Then slow drift in distribution across circles governed by

$$\frac{\partial H}{\partial T} = -\frac{1}{6\pi}\int_0^{2\pi} \frac{1}{r}\frac{\partial}{\partial r}\left[rH^3\nabla\kappa\right]_{\perp r}\, d\theta,$$

and first correction given by

$$h_1 = \frac{1}{3r(1+X'\cos\theta)}\Bigg(\sqrt{1+2X'\cos\theta+X'^2}\left[H^3\nabla\kappa\right]_{\perp\theta} + \int_0^{\theta}\frac{\partial}{\partial r}\left[rH^3\nabla\kappa\right)_{\perp r}\, d\theta\Bigg).$$

Numerics

Central space differencing, awkward for non-orthogonal cross derivatives.

Explicit time-step, stability if

$$\delta T < \tfrac{1}{20}\left(\delta r \min_r (1 - X')\right)^4$$

but independent of $\delta\theta$.

Typically, for $\gamma = 0.2$,

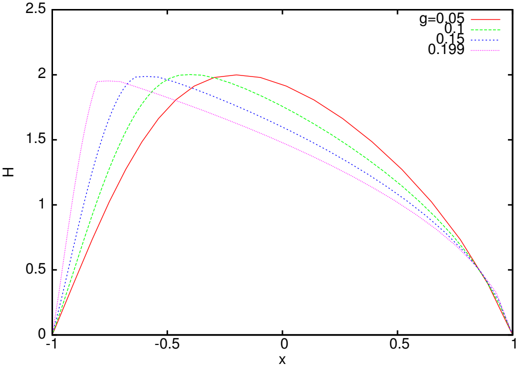
$$\delta r = \delta\theta = 0.05, \quad \delta T = 10^{-10}$$

but attains equilibrium by $T = 0.005$.

Problem that circles intersect, $X' > 1$, for initial condition $H = 2(1 - r^2)$ when $\gamma > 0.16$. Hence must ramp up γ in time.

Results

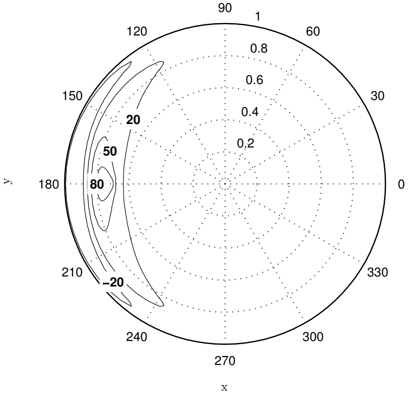
Profiles of the coating on $y = 0$ for various γ



Position of maximum h moving to $(-1, 0)$.

Results

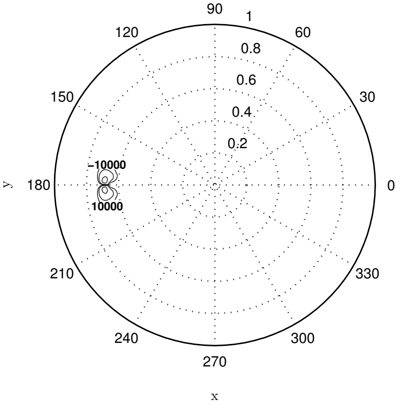
Capillary pressure at $\gamma = 0.15$



due to high curvature as maximum h pushed to edge $x = -1$.

Results

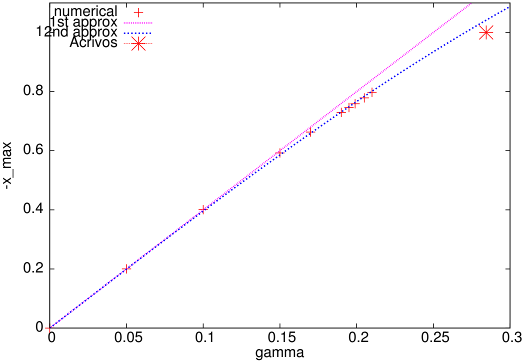
$O(\alpha)$ correction $h_1(r\theta)$ at $\gamma = 0.15$. Extreme values $\pm 9\,10^4$



Thicker below $y = 0$ where gradient of capillary pressure opposes rotation, like gravity (clockwise rotation).

Results

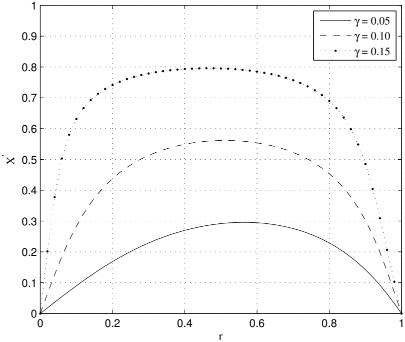
Position of maximum thickness as γ varies



Numerical data to $\gamma = 0.210$,
asymptotics 1st order -4γ, 2nd order $-(2 - \frac{15}{16}\gamma)^2\gamma$,
and Acrivos's speculation $\gamma_{\max} = 0.2844$.

Results

Separation of circles on $y = 0$ is $1 - X'$



If $X' \to 1$ as $\gamma \to \gamma_{\max}$ with $X = -\gamma H^2$, then Acrivos

$$H \to \frac{1}{\sqrt{\gamma_{\max}}} \sqrt{1 - r}$$