Instabilities of a thin coating on a vertical fibre; Newtonian, shear-thinning, and elastic liquids

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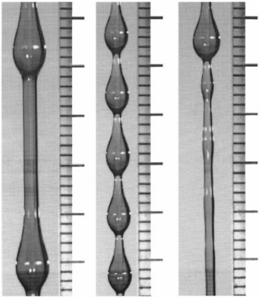
and Claire McIlroy for elastic liquids

Motivation

Manufacture of polymeric and optical fibres.



Newtonian

Kliakhandler, Davis & Bankoff JFM 2001

Shear-thinning

Duprat, Ruyer-Quil & Giorgiutti-Dauphiné

Phys. Fluids 2009

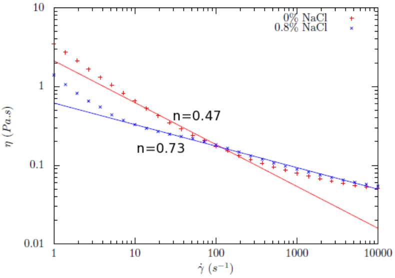
The coating fluid is often non-Newtonian

Governing equations

Constitutive equation

Power-law viscosity: $\mu = \beta \left| \dfrac{\partial u}{\partial y} \right|^{n-1}$

Xanthan solutions



Boulogne et al, Private Communication

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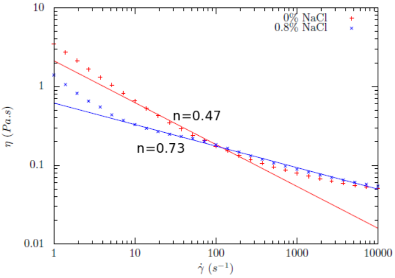
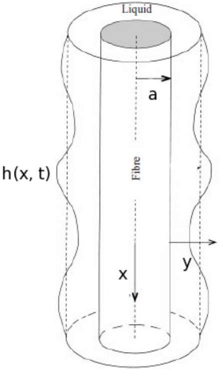


Boulogne et al, Private Communication

Geometry
Axisymmetric



Governing equations

Lubrication framework

Capillary pressure: $$p = -\gamma \left(\frac{h}{a^2} + h_{xx} \right)$$

Momentum: $$0 = -\frac{\mathrm{d}p}{\mathrm{d}x} + \rho g + \frac{\partial \sigma_{xy}}{\partial y}$$

Volume flux: $$Q = \beta^{-\frac{1}{n}} \frac{n}{2n+1} \left(\rho g - \frac{\mathrm{d}p}{\mathrm{d}x} \right)^{\frac{1}{n}} h^{(2+\frac{1}{n})}$$

Note: $(\cdot)^{\frac{1}{n}} = \mathrm{sign}(\cdot)|\cdot|^{\frac{1}{n}}$

Mass conservation: $$h_t + Q_x = 0$$

Lengthscales:

- Fibre radius, a, in x direction.
- Initial film thickness, h_0, in y direction.

Time:

- Rayleigh instability, $\frac{2n+1}{n}\left(\frac{\beta a^{n+3}}{\gamma h_0^{n+2}}\right)^{\frac{1}{n}}$.

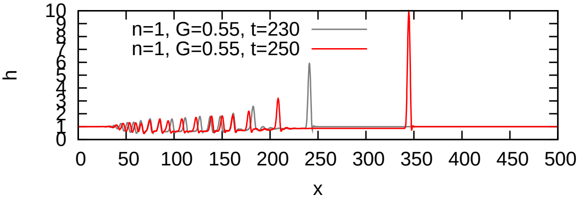
$$\boxed{h_t + \left(h^{2+\frac{1}{n}}(G + (h + h_{xx})_x)^{\frac{1}{n}}\right)_x = 0}$$

where Bond number $G = \frac{\rho g a^3}{\gamma h_0}$.

Time-dependent numerical simulations

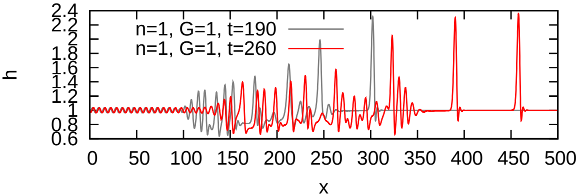
Periodic forcing at inlet: $\omega = 1$

G small (thicker film):



Ever growing large pulses

G big (thinner film):



Solitary waves of similar amplitude and speed

This talk: Solitary waves? When? Properties?

In the frame of the solitary waves travelling with speed c:

$$(G + (h + h_{xx})_x) = \frac{\left(c(h-1) + G^{\frac{1}{n}}\right)^n}{h^{2n+1}}$$

$$h \to 1, \quad \text{as} \quad x \to \pm\infty$$

Numerically construct the stationary solitary waves.

- Integrate from $x = -\infty$ to $x = 0$,
  and from $x = +\infty$ to $x = 0$.
- Hence need starting conditions at $x = \pm\infty$.

Stationary solitary waves

Initial conditions for numerics

At $x = \pm\infty$: $h \sim 1 + \tilde{h}$ with $\tilde{h} \ll 1$.
Linearised equation:

$$\tilde{h}''' + \tilde{h}' - A\tilde{h} = 0$$

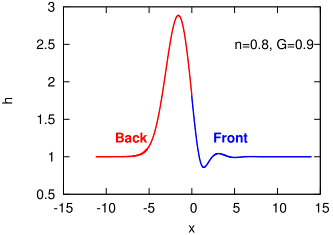
where $A = nG^{1-1/n}c - (2n+1)G > 0$.

Three solutions of exponential form:

- $\tilde{h}_1 = a_1 e^{m_1 x}$
  m_1 real and positive: growing mode.

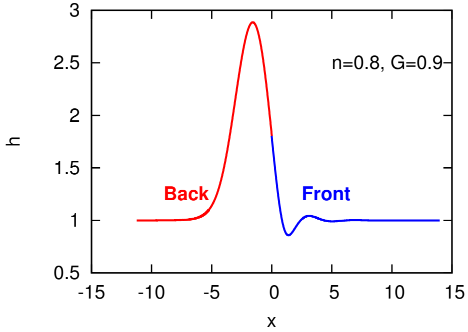
  Use in 'Back' (1 DoF).
- $\tilde{h}_{2,3} = a_{2,3} e^{m_{2,3} x}$
  $m_{2,3}$ complex conjugates with negative real part: decaying modes.

  Use in 'Front' (2 DoF).



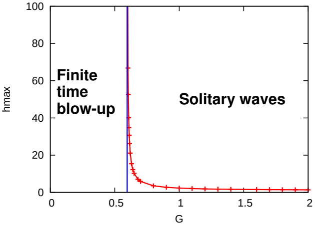
For fixed G:

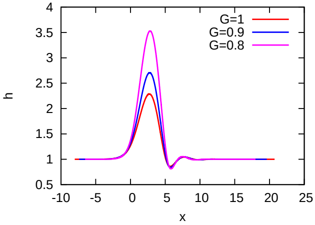
1. Shoot from Back, with $a_2 = a_3 = 0$. Stop when $h'' = 0$, $h' < 0$.
2. Shoot from Front, with $a_1 = 0$. Stop when $h'' = 0$, $h' < 0$, $h > 1.5$.
3. Vary the phase of $a_{2,3}$ in Front to match h.
4. Vary speed c to match h'.



Stationary solitary waves

Results: $n = 1$ Kalliadasis & Chang, J. Fluid Mech. 1994





- No stationary solitary waves for $G < G_0$.
- As $G \downarrow G_{0+}$, $h_{\max} \to \infty$.

- Width of the 'Main Body' independent of G.

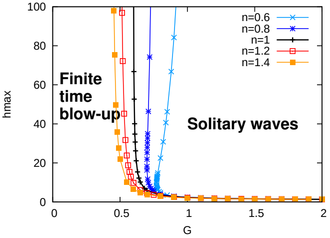
Agreement with experiment Quéré, Europhys. Lett. 1990:

- Critical h_c to observe disturbance $\propto a^3$.
- $G = \frac{\rho g a^3}{\sigma h_0} \Rightarrow h_c \propto a^3$ at $G = G_0$.

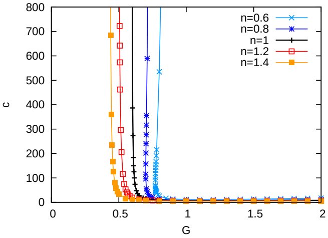
Stationary solitary waves

Results: various n (shear-thinning and shear-thickening)

$h_{\max}$ vs G



c vs G



- Two branches of solutions for $n < 1$.

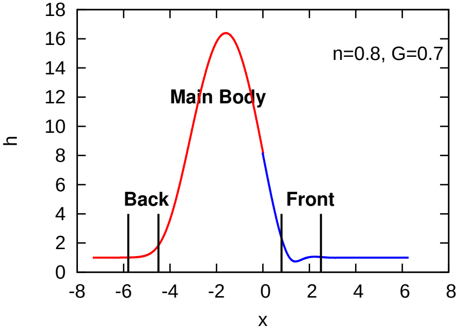
Look at large fast stationary solitary waves close to G_0.

What determines critical G_0? Relationship of h and c with G?

Large fast solitary waves

Pulse divided into 3 regions:

- 'Main body' region: h big, $x \sim O(1)$.
- 'Front' and 'Back' transition regions: $h \sim O(1)$, x small.



Asymptotic analysis for each region, and match. Very complicated!

Large fast solitary waves

Main body region: leading order

h big, $x \sim O(1)$

$$(G + (\mathbf{h} + \mathbf{h_{xx}})_{\mathbf{x}}) = \frac{\left(c(h-1) + G^{\frac{1}{n}}\right)^n}{h^{2n+1}}$$

Solution: constant capillary pressure ($p = \frac{1}{2} h_{\max}$)

$$h = \frac{1}{2} h_{\max} (1 - \cos x) \quad \text{in} \quad 0 \le x \le 2\pi.$$

For matching,

$$h \sim \frac{1}{4} h_{\max} (x - x_0)^2,$$

with $x_0 = 0$ at the Back and $x_0 = 2\pi$ at the Front.

At leading order main body is at a constant pressure

Large fast solitary waves

Transition regions: leading order

$h \sim O(1)$, x small

$$(G + (h + \mathbf{h_{xx}})_{\mathbf{x}}) = \frac{\left(c(\mathbf{h} - \mathbf{1}) + G^{\frac{1}{n}}\right)^{n}}{\mathbf{h^{2n+1}}}$$

Transition regions: $x \sim c^{-n/3}$.

Modified Bretherton equation:

$$h_{\xi\xi\xi} = \frac{(h-1)^n}{h^{2n+1}} \quad \text{with} \quad \xi = c^{n/3}(x - x_0).$$

$(x_0 = 0$ at 'Back' and $x_0 = 2\pi$ at 'Front'.)

$h \sim O(1)$, x small

$$\left(G + (h + \mathbf{h_{xx}})_{\mathbf{x}}\right) = \frac{\left(c(\mathbf{h} - \mathbf{1}) + G^{\frac{1}{n}}\right)^{n}}{\mathbf{h^{2n+1}}}$$

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($x_0 = 0$ at 'Back' and $x_0 = 2\pi$ at 'Front'.)

For matching, solutions towards 'Main Body' (h becoming large)

$$h \sim \tfrac{1}{2}P_{\pm}\xi^2 + Q\xi + R_{\pm} \quad \text{as} \quad \xi \to \pm\infty$$

Use 1 DoF to redefine origin so $Q = 0$.

Large fast solitary waves

Matching: leading order

DoFs at Back: $1-1(Q=0)=0$. P_+ and R_+ uniquely determined.
DoFs at Front: $2-1(Q=0)=1$. One parameter in P_- and R_-.

Main body: $h \sim \frac{1}{4}h_{\max}(x-x_0)^2 \quad \text{near} \quad x_0 = 0,\, 2\pi.$

Transition regions: $h \sim \frac{1}{2}P_\pm \xi^2 + R_\pm \quad \text{as} \quad \xi \to \pm\infty.$

Matching:

$$P_- = P_+$$

So now P_- unique and hence R_- unique.

$$\tfrac{1}{2}P(\xi = c^{n/3}(x-x_0))^2 = \tfrac{1}{4}h_{\max}(x-x_0)^2$$

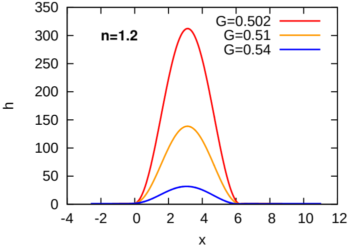
$$\boxed{h_{\max} = 2Pc^{2n/3}}$$

Note: capillary pressure in the main body $p = \frac{1}{2}h_{max} = Pc^{2n/3}$.

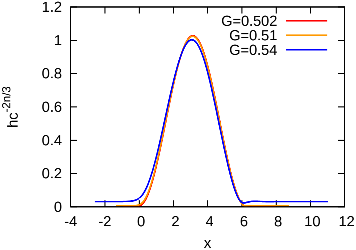
Large fast solitary waves

Checking scalings

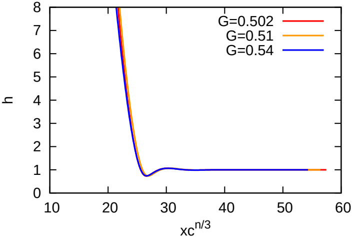
Original profile:



Main body:
height scaled by $c^{-2n/3}$



'Front' transition region:
width scaled by $c^{n/3}$



Large fast solitary waves

So far have $h_{\max}(c)$. G yet to appear

Transition regions: $h \sim \frac{1}{2}P\xi^2 + R_\pm$.

- Different apparent film thickness, $R_\pm$, at 'Back' and 'Front'.

Need 1st correction of Main Body: $h \sim c^{2n/3}h_0 + h_2$

$$(\mathbf{G} + (\mathbf{h} + \mathbf{h_{xx}})_\mathbf{x}) = \frac{\left(c(h-1) + G^{\frac{1}{n}}\right)^n}{h^{2n+1}}$$

$$G_0 + (h_2 + h_{2xx})_x = 0$$

Large fast solitary waves

So far have $h_{\max}(c)$. G yet to appear

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$$(\mathbf{G} + (\mathbf{h} + \mathbf{h_{xx}})_\mathbf{x}) = \frac{\left(c(h-1) + G^{\frac{1}{n}}\right)^n}{h^{2n+1}}$$

$$G_0 + (h_2 + h_{2xx})_x = 0$$

Solution (hydrostatic pressure gradient):

$$h_2 = -G_0(x - \sin x) + R_+ \quad \text{in} \quad 0 \le x \le 2\pi.$$

Matching gives critical G_0:

$$\boxed{G_0 = (R_+ - R_-)/2\pi}$$

$2\pi G_0$ pressure difference between pushing and pulling transitions

Large fast solitary waves

c as a function of G

So far have $h_{\max}(c)$ and critical G_0. Yet to find $G(c)$.

Need 2nd correction:

$$h \sim c^{2n/3} h_0 + h_2 + c^{-(2n-1)n/3} h_3$$

$$G = G_0 + c^{-(2n-1)n/3} G_1$$

$$(h_3 + h_{3xx})_x = \left(\frac{1}{P^{n+1}(1-\cos x)^{n+1}} - G_1 \right)$$

Solution

$$\begin{aligned} P^{n+1} h_3 = & \frac{(n+1)\sin x}{n(2n+1)(1-\cos x)^n} - \frac{(n+(n+1)\cos x)\sin x}{(2n+1)(2n-1)(1-\cos x)^n} \\ & + \frac{(n-1)(n+(n+1)\cos x)}{(2n+1)(2n-1)} + \int_\pi^x \frac{1}{(1-\cos t)^{n-1}}\, dt - G_1 x \end{aligned}$$

Near $x = x_0$

$$h_3 \sim S(x - x_0)^{1-2n} + D_\pm - G_1 x + \ldots$$

- The singular term matches the same in transition regions.
- $D_\pm$ different at the 'Back' and 'Front'.
- No terms to match with them from transition regions.
- Hence need:

$$G_1 = (D_+ - D_-)/2\pi$$

Finally we have found the relationship between c and G

$$\boxed{G = G_0 + c^{-(2n-1)n/3} G_1}$$

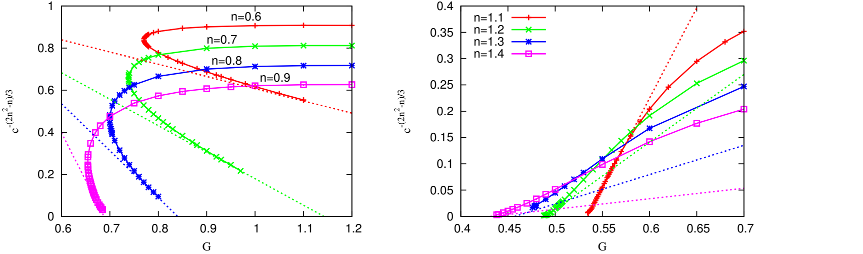
$2\pi G_1$ is the extra pressure difference to drive flow through main body compared with $n = 1$

Large fast solitary waves

Results

$$\boxed{G = G_0 + c^{-(2n-1)n/3} G_1}$$

Plot $c^{-(2n-1)n/3}$ vs G



- When $n < 1$, $G_1 < 0$. Negative slope at G_0.
- When $n > 1$, $G_1 > 0$. Positive slope at G_0.
- When $n = 1$, $G_1 = 0$. No relationship between G and c yet.

More terms

transition regions

With scaling $\xi = c^{n/3}(x - x_0)$,

$$h_{\xi\xi\xi} = \frac{(h-1)^n}{h^{2n+1}} - c^{-2n/3} h_\xi - c^{-n} G + c^{-1} \frac{n(h-1)^{n-1} G^{1/n}}{h^{2n+1}} + \dots$$

Expand h as

$$h \sim h_0 + c^{-2n/3} h_2 + c^{-n} h_3 + c^{-1} h_4 + c^{-4n/3} h_5 + \dots$$

$$h_0''' = \frac{(h-1)^n}{h^{2n+1}},$$

$$h_2''' = \frac{(h_0-1)^{n-1}\left(-(n+1)h_0 + (2n+1)\right)}{h_0^{2n+2}} h_2 - \mathbf{h_0'},$$

$$h_3''' = \frac{(h_0-1)^{n-1}(-(n+1)h_0 + (2n+1))}{h_0^{2n+2}} h_3 - \mathbf{G_0},$$

$$h_4''' = \frac{(h_0-1)^{n-1}(-(n+1)h_0 + (2n+1))}{h_0^{2n+2}} h_4 + \frac{n(h_0-1)^{n-1} G_0^{1/n}}{h_0^{2n+1}},$$

$$h_0 \sim \frac{P}{2}\xi^2 + \mathbf{R}_\pm + Sx^{1-2n}$$

$$h_2 \sim -\frac{P}{4!}\xi^4 + \frac{a_{2\pm}}{2}\xi^2 + \mathbf{c_{2\pm}} + k_2\xi^{3-2n}$$

$$h_3 \sim -\frac{G_0}{3!}\xi^3 + \frac{a_{3\pm}}{2}\xi^2 + \mathbf{c_{3\pm}}$$

$$h_4 \sim \frac{1}{2} a_{4\pm}\xi^2 + c_{4\pm}$$

More terms

main body

With $h = c^{2n/3}H$,

$$(H + H_{xx})_x = -c^{-2n/3} G + c^{-(2n+1)n/3} \frac{(1 - \frac{c^{-2n/3}}{H} + \frac{G^{1/n}(c^{-1-2n/3})}{H})^n}{H^{n+1}}.$$

Expand H as

$$H \sim H_0 + c^{-2n/3} H_2 + c^{-(2n+1)n/3} H_3 + c^{-n} H_4 + c^{-1} H_5 + c^{-4n/3} H_6 + \dots$$

and G as

$$G \sim G_0 + G_1 c^{-(2n-1)n/3} + G_2 c^{-2n/3} + \dots$$

$$\begin{aligned}
H_0' + H_0''' &= 0, & H_0 &= P(1 - \cos x) \\
H_2' + H_2''' &= -G_0 & H_2 &= G_0 (\sin x - x) + A_2 + C_2 \cos x \\
H_3' + H_3''' &= -G_1 + \frac{1}{P^{n+1}(1 - \cos x)^{n+1}}, & H_3 &\sim S x^{1-2n} + D_{\pm} - G_1 x + k_2 x^{3-2n} \\
H_4' + H_4''' &= 0, & H_4 &= A_4 + B_4 \sin x + C_4 \cos x \\
H_5' + H_5''' &= 0, & H_5 &= A_5 + B_5 \sin x + C_5 \cos x \\
H_6' + H_6''' &= -G_2, & H_6 &= G_2(\sin x - x) + A_6 + B_6 \sin x + C_6 \cos x
\end{aligned}$$

Transition regions=

$$
\begin{array}{rllllll}
 & h_0 & h_2 & h_3 & h_4 & h_5 & \\
c^{\frac{2n}{3}}\Big[& \frac{P}{2}x^2 & -\frac{P}{4!}x^4 & & & +\frac{P}{6!}x^6 & +\dots\Big] \\
+c^0\Big[& R_\pm & +\frac{a_2}{2}x^2 & -\frac{G_0}{3!}x^3 & & -\frac{a_2}{4!}x^4 & +\dots\Big] \\
+c^{-\frac{2n^2}{3}+\frac{n}{3}}\Big[& Sx^{1-2n} & +k_2x^{3-2n} & & & +k_3x^{5-2n} & +\dots\Big] \\
+c^{-\frac{n}{3}}\Big[& & & +\frac{a_3}{2}x^2 & & & +\dots\Big] \\
+c^{\frac{2n}{3}-1}\Big[& & & & \frac{a_4}{2}x^2 & & +\dots\Big] \\
+c^{-\frac{2n}{3}}\Big[& & c_{2\pm} & & & +\frac{a_5}{2}x^2 & +\dots\Big]
\end{array}
$$

Main body=

$$
\begin{aligned}
& c^{\frac{2n}{3}} && \left[\tfrac{P}{2}x^2 - \tfrac{P}{4!}x^4 + \tfrac{P}{6!}x^6 + \dots\right] \\
& +c^{0} && \left[-G_0x_0 + A_2 + C_2 - \tfrac{C_2}{2}x^2 - \tfrac{G_0}{3!}x^3 + \tfrac{C_2}{4!}x^4 + \dots\right] \\
& +c^{-\frac{2n^2}{3}+\frac{n}{3}} && \left[Sx^{1-2n} - G_1x_0 + D_{\pm} + k_2x^{3-2n} + k_3x^{5-2n} + \dots\right] \\
& +c^{-\frac{n}{3}} && \left[A_4 + C_4 - \tfrac{C_4}{2}x^2 + \dots\right] \\
& +c^{\frac{2n}{3}-1} && \left[A_5 + C_5 - \tfrac{C_5}{2}x^2\right] \\
& +c^{-\frac{2n}{3}} && \left[-G_2x_0 + A_6 + C_6 - \tfrac{C_6}{2}x^2 - \tfrac{G_2+B_6}{3!}x^3 + \dots\right]
\end{aligned}
$$

More terms: matching two regions

At c^0:

$$G_0 = (R_+ - R_-)/2\pi$$

At $c^{-(2n^2-n)/3}$:

$$G_1 = -(D_+ - D_-)/2\pi$$

At c^{-1}:

$$G_2 = (c_{2+} - c_{2-})/2\pi$$

Hence,

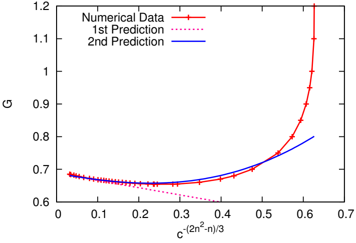
$$\boxed{G = G_0 + G_1 c^{-(2n-1)n/3} + G_2 c^{-2n/3}}$$

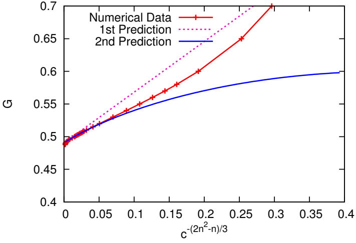
More terms: Results

$$\boxed{G = G_0 + G_1 c^{-(2n-1)n/3} + G_2 c^{-2n/3}}$$

Plot G vs $c^{-(2n-1)n/3}$

$n = 0.9$ $n = 1.2$





More terms: Results

$$\boxed{G = G_0 + G_1 c^{-(2n^2-n)/3} + G_2 c^{-2n/3}}$$

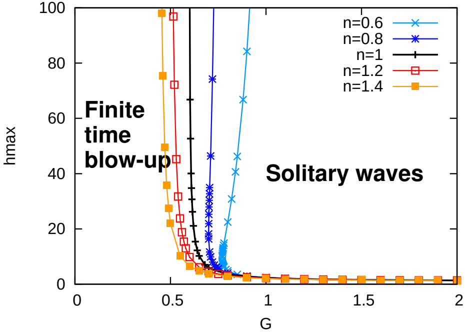
When $n = 1$, $G_1 = 0$, so

$$G = G_0 + G_2 c^{-2/3}$$

Need even more terms for Newtonian $n = 1$ – see beyond end.

$2\pi G_2$ comes from corrections in the transition regions due to the small axial curvature

Two branches for $n < 1$



Upper branch is unstable – solutions either blow up or decay to lower branch.

Hence there is a maximum size of stable solitary for shear-thinning fluids.

Future Work

- Normal stress effect. $\surd$
- Relax the thin film approximation? $\surd$

  A third mechanism to determine $G(c)$
- Newtonian fluid $n = 1$ $\surd$
- What happens at big G? $\surd$
- Finite flow domain for shear-thinning fluids $\surd$
- Comparison with experimental data.

Symmetry breaking instability with elastic liquids

with Claire McIlroy

- François Boulogne observed in his Paris PhD thesis that the coating of an elastic liquid was never axisymmetric, but was always thicker on one side.
- Flow in thin coating is mainly simple shear and quasi-steady (varies over distances much greater than thickness).
- Hence rheology is a viscosity plus normal stresses.
- First normal stress difference = tension in streamlines $\rightarrow$ enhanced effective surface tension.
- Second normal stress difference = tension in vortex lines $\rightarrow$ new instability.

Symmetry breaking instability with elastic liquids

Governing equation

Extra non-Newtonian stress for a second-order fluid

$$\sigma^{NN} = -2\alpha \overset{\triangledown}{E} + \beta E^2,$$

α tension in the streamlines, $\beta < 0$ tension in the vortex lines.

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial z} - \nabla h^3 \nabla \nabla^2 h - A \frac{\partial^2}{\partial z^2} h^5 - B \frac{\partial^2}{\partial \theta^2} h^5 = 0,$$

(curiously $A \sim \alpha/6$, but $B \sim -\beta/80$)

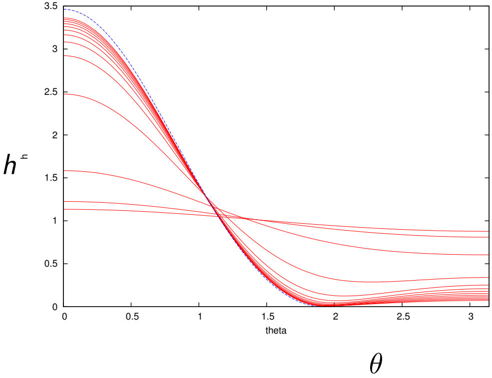
Now study development of lop-sided flow with $h(\theta, t)$, no z-variations.

$$h_t - \left(h^3(h_{\theta\theta} + h + Bh^2)_\theta\right)_\theta = 0$$

Symmetry breaking instability with elastic liquids

Time evolution

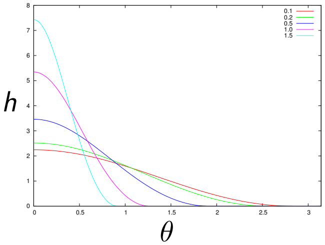
$h(\theta, t)$ at $t = 2^n$ $n = -2, \ldots, 11$, for $B = 0.5$.



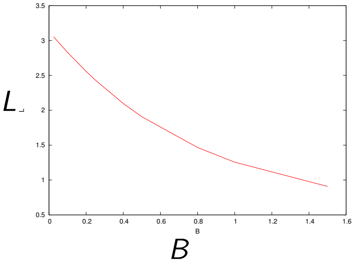
Dotted blue is a steady state which wets only $0 \leq \theta \leq 1.9071$

(Interesting intermediate times: drift of an off-centred cylinder.)

Steady states for various B



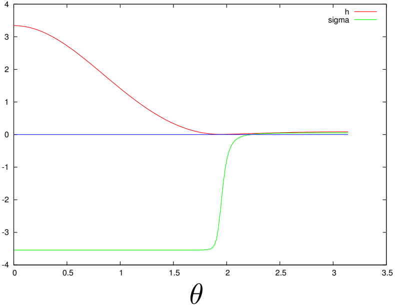
Length of steady state



Symmetry breaking instability with elastic liquids

Structure of steady states

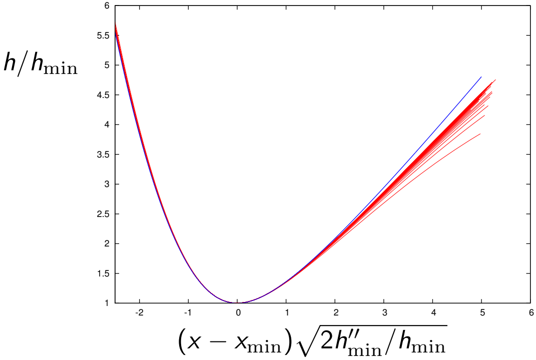
The shape and the pressure (stress $\sigma_{\theta\theta}$) at $t = 10^3$ for $B = 0.5$



There two constant pressure regions.

The higher pressure region to the right is draining into the lower pressure region to the left through a small neck.

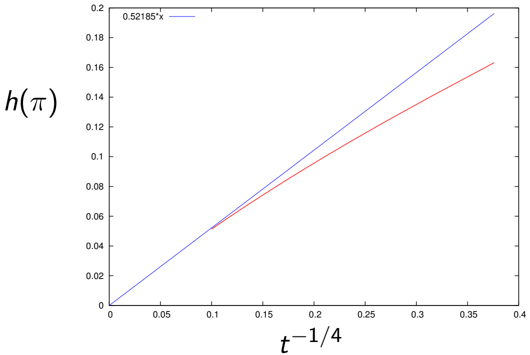
Universal shape of the neck between the two constant pressure regions, for $t = 50\,(50)\,10^3$ and for $B = 0.5$.



Blue shape from Bretherton's equation.

Symmetry breaking instability with elastic liquids

Draining of small region



$$h(\pi) = \frac{1+\cos L}{t^{1/4}} \left(\frac{K((\pi - L)\cos L + \sin L}{4Q\sin^5 L} \right)^{1/4}$$

with Bretherton $Q = 1.20936$ and for $B = 0.5$ pressure in steady state lobe $K = 3.7297$ and length of steady state lobe $L = 1.9171$.

Future Work

- Normal stress effect. $\surd$
- Relax the thin film approximation? $\surd$

  A third mechanism to determine $G(c)$
- Newtonian fluid $n = 1$ $\surd$
- What happens at big G? $\surd$
- Finite flow domain for shear-thinning fluids $\surd$
- Comparison with experimental data.

$n = 1$ Newtonian fluid, even more terms

Matching: transition regions

Transition regions=

$$
\begin{array}{rlllll}
 & h_0 & h_2 & h_3 & h_4 & \\
c^{2/3}\Big[& \frac{P}{2}x^2 & -\frac{P}{4!}x^4 & & +\frac{P}{6!}x^6 & +\dots\Big] \\
+c^0\Big[& R_\pm & +\frac{a_2}{2}x^2 & -\frac{G_0}{3!}x^3 & -\frac{a_2}{4!}x^4 & +\dots\Big] \\
+c^{-1/3}\Big[& -\frac{2}{3P^2x} & & +\frac{a_3}{2}x^2 & +\frac{11}{1080P^2}x^3 & +\dots\Big] \\
+c^{-2/3}\Big[& & +c_{2\pm} & & +\frac{a_4}{2}x^2 & +\dots\Big] \\
+c^{-1}\log c\Big[& & & +\frac{4G_0}{9P^3} & & +\dots\Big] \\
+c^{-1}\Big[& \frac{2(1+2R_\pm)}{15P^3x^3} & +\frac{8R_\pm+4+20a_2}{15P^3x} & +\frac{4G_0}{3P^3}\log x + c_{3\pm} & & +\dots\Big]
\end{array}
$$

$n = 1$ Newtonian fluid, even more terms

Matching: main body region

Main body =

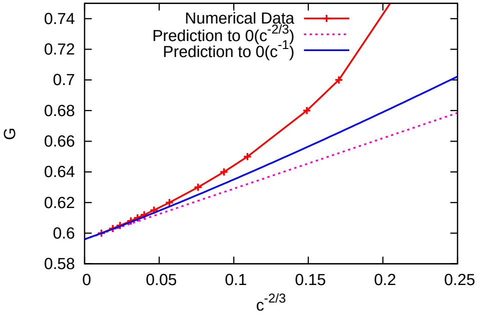
$$
\begin{aligned}
& c^{2/3} && \left[\tfrac{P}{2}x^2 - \tfrac{P}{4!}x^4 + \tfrac{P}{6!}x^6 + \dots\right] \\
& +c^0 && \left[-G_0 x_0 + A_2 + C_2 - \tfrac{C_2}{2}x^2 - \tfrac{G_0}{3!}x^3 + \tfrac{C_2}{4!}x^4 + \dots\right] \\
& +c^{-1/3} && \left[-\tfrac{2}{3P^2 x} + (A_3 + C_3) + (\tfrac{1}{18P^2} + B_3)x - \tfrac{C_3}{2}x^2 + (\tfrac{1}{1080P^2} - \tfrac{B_3}{3!})x^3 \dots\right. \\
& +c^{-2/3} && \left[-G_2 x_0 + A_4 + C_4 + B_4 x - \tfrac{C_4}{2}x^2 - \tfrac{G_2}{3!}x^3 + \dots\right] \\
& +c^{-1}\log c && \left[A_5 + C_5 - \tfrac{C_5}{2}x^2 + \dots\right] \\
& +c^{-1} && \left[\tfrac{2(1+2R_\pm)}{15P^3x^3} + \tfrac{4(1+2A_2-3C_2)}{15P^3x} + \tfrac{4G_0}{3P^3}\log x - G_3 x_0 + A_6 + C_6 \dots\right]
\end{aligned}
$$

$n = 1$ Newtonian fluid, even more terms

Results

At c^0: $\quad G_0 = (R_+ - R_-)/2\pi$
At $c^{-2/3}$: $\quad G_2 = (c_{2+} - c_{2-})/2\pi$
At c^{-1}: $\quad G_3 = (c_{3+} - c_{3-})/2\pi$
Hence,

$$\boxed{G = G_0 + G_2 c^{-2/3} + G_3 c^{-1}}$$



Big G

$$h \sim 1 + \frac{1}{G} h_1 \quad c \sim \left(2 + \frac{1}{n}\right) G^{\frac{1}{n}} + c_1 G^{\frac{1}{n}-1}$$

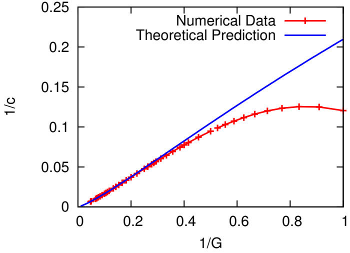
where h_1 satisfies the nonlinear equation

$$h_1' + h_1''' = n c_1 h_1 + h_1^2 \left(-n(2n+1) + \frac{n(n-1)}{2} \left(2 + \frac{1}{n}\right)^2 \right)$$

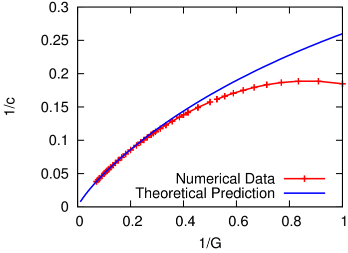
This equation can be solved numerically to give the value of c_1 for different values of n.

Big G results

$n = 0.8$



$n = 1.2$



Finite flow domain for shear-thinning fluids

Modified Bretherton equation

$$h''' = \frac{(h-1)^n}{h^{2n+1}}$$

Integrating from $\pm\infty$ where $h \sim 1 + \tilde{h}$ ($\tilde{h} \ll 1$), $\tilde{h}$ satisfies:

$$\tilde{h}''' = \tilde{h}^n. \Leftarrow \text{ No exponential solutions for } n \neq 1.$$

Solution at 'Back'

$$\tilde{h} = A(\xi - \xi_0)^{\frac{3}{1-n}}, \quad n < 1$$

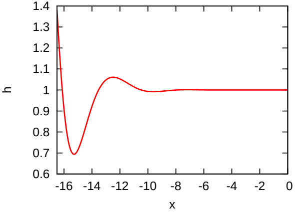
$\tilde{h}$ becomes 0 at a finite distance.

While viscosity thins as $\gamma \to \infty$ it thickens as $\gamma \to 0$, and so flow stops in a finite distance.

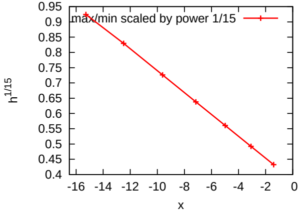
Finite flow domain for shear-thinning fluids

Solution at 'Front' ($n = 0.8$)

Decaying nonlinear oscillations

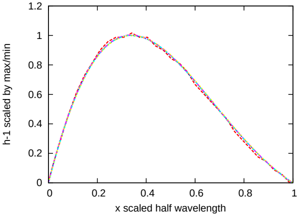


$h_{\max}^{3/(1-n)}$ vs $x_{\max}$



Decays to zero in finite distance

Each half-cycle normalised by maximum and by wavelength



Universal shape