Steady streaming in the formation of sand ripples

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with Emeline Larrieu

DAMTP/Cambridge & IMFT/Toulouse

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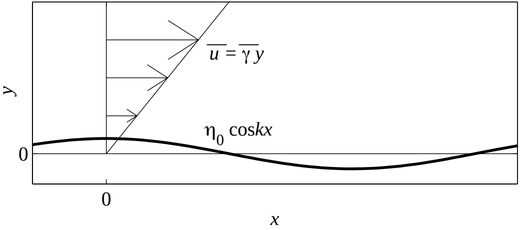
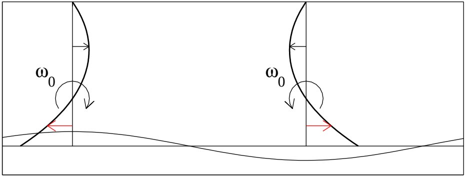
Two-layer shear instability?

Sand bed = very viscous liquid?



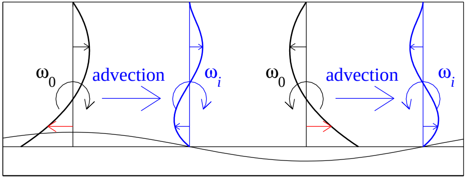
Instability due to jump in viscosity/velocity gradient (≈KH)

Mechanism



No slip on perturbed surface

Mechanism

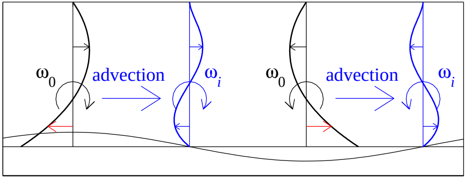


No slip on perturbed surface

Advection of vorticity

Induced flow from trough to crest

Mechanism



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Stabilised by

- adverse gravity going up to crest

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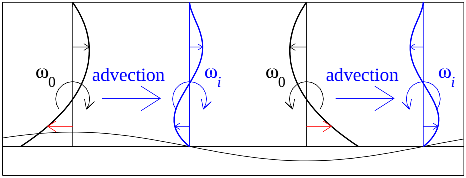

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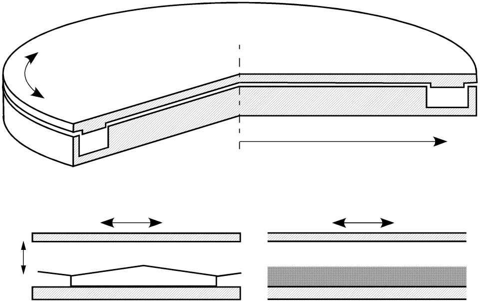
Advection of vorticity

Induced flow from trough to crest

Stabilised by

- adverse gravity going up to crest
- erosion from crest, depositing in troughs

Experiments in annulus at IMFT by Hélène Mouilleron



$R = 200\,\mathrm{mm}$,
$h = 15\ \&\ 7\,\mathrm{mm}$,
$\eta_0 = 1\,\mathrm{mm}$,
$\lambda = 50\&70\,\mathrm{mm}$,
$d = 0.5\,\mathrm{mm}$,
$\mu = 0.05\ \&\ 1\,\mathrm{Pa\,s}$.

Matched optical index of particles and fluid

Small secondary circulation due to centrifugal force

Experiments in annulus at IMFT by Hélène Mouilleron

Instability not seen in steady experiments

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Instability not seen in steady experiments

But seen in oscillating flow

Experiments in annulus at IMFT by Hélène Mouilleron

Instability not seen in steady experiments

But seen in oscillating flow

because erosion from crests suppressed.

A diversion: Non-erosion of crests

Does not happen for standard model of bedload transport:

$$\text{flux} \quad q(x,t) = C(\gamma^n(x,t) - \gamma_c^n), \quad \text{surface shear rate } \gamma.$$

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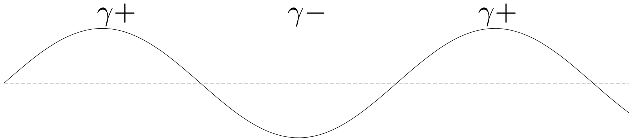

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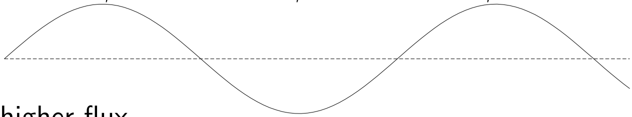
$\gamma+$ $\quad$ $\gamma-$ $\quad$ $\gamma+$



Hence higher flux

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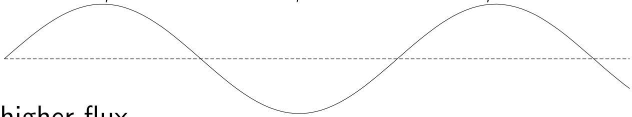

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Divergence of flux

deposit loss

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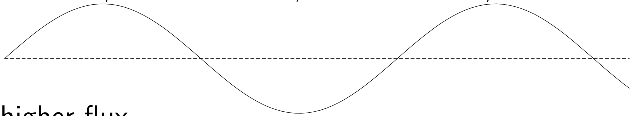

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Divergence of flux

deposit $\qquad$ loss

Hence wave propagates to right without growth or decay

Erosion-deposition model for erosion of crests

Surface density of mobile grains $n(x,t)$.

$$\frac{\partial n}{\partial t} + \frac{\partial q}{\partial x} = -\frac{1}{\tau_{\mathrm{sed}}} n + \frac{1}{d^2}(\gamma - \gamma_c) \qquad \text{with} \quad q = \gamma d n.$$

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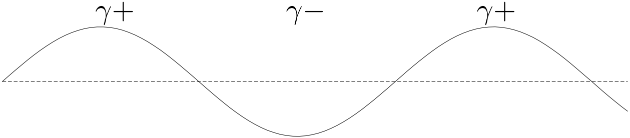

On crests: $\gamma+$ produces $n+$, produces $q+$.

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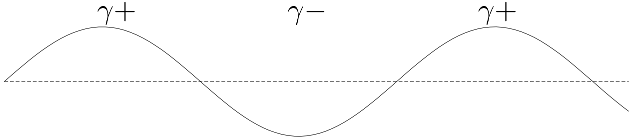

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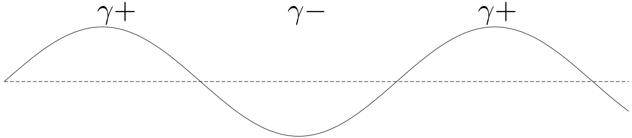

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In troughs: Div δq deposits to fill trough, similarly erode crest.

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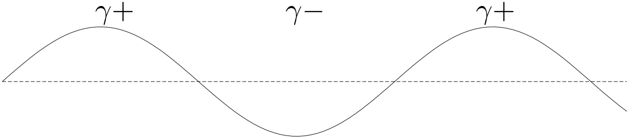

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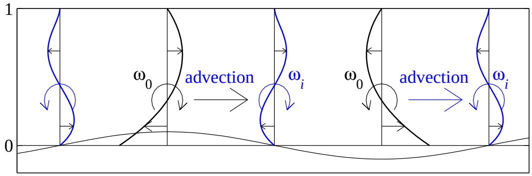
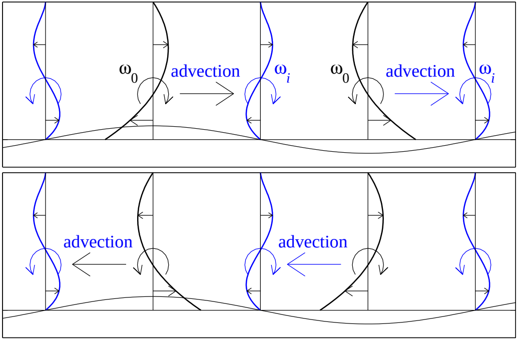
Small displacements in oscillating flow, reduce erosion by $1/(\omega\tau)^2$.

Back to instability mechanism, now in oscillating flow



Flow to right

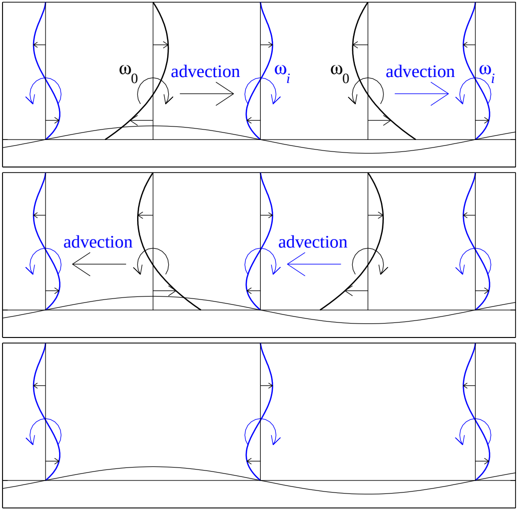
Mechanism in oscillating flow



Flow to right

Flow to left

Mechanism in oscillating flow



Flow to right

Flow to left

Steady streaming

Steady streaming from troughs to crests is mechanism in all regimes of oscillation flows.

Calculation of steady streaming

For experimental conditions, not sea conditions.

Calculation of steady streaming

In general, there are 7 lengths:

- d particle diameter
- η_0 amplitude of ripples,
- λ wavelength of ripples. Vortices shed if $\eta_0 > 0.1\lambda$
- δ thickness of Stokes oscillation boundary layer, $\delta = \sqrt{\nu/\omega}$
- ℓ excursion of fluid in oscillating flow
- h depth of layer
- L wavelength of water waves

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If $d \ll \eta_0$, d only in particle transport equation.

Sea ripples: $\lambda, \delta \ll h, L$

IMFT experiments: $\eta_0 \ll h \ll \delta, \lambda$

Calculation of steady streaming

For experimental conditions, not sea conditions.

- Thin layer, $kh \ll 1$.
  - $O(1)$ term only.
- Small disturbance $\epsilon = \eta_0/h \ll 1$.
  - $O(1)$ flat-bottom and
  - $O(\epsilon)$ first effect of wavy-bottom.
- Small Reynolds number, $Re = \rho\omega h^2/\mu \ll 1$.
  - $O(1)$ Stokes flow and
  - $O(Re)$ first inertial correction.
- Amplitude $A = kU_0/\omega$. Two cases small and $O(1)$.

Non-dimensionalised governing equations

Thin-layer (boundary layer) approximation for horizontal velocity

$$Re\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{dp}{dx} + \frac{\partial^2 u}{\partial y^2}.$$

Vertical velocity from

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

(Also gives pressure so that horizontal flux is constant.)

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Flat top

$$\mathbf{u} = (A\cos t, 0) \quad \text{on } y = 1,$$

Wavy bottom

$$\mathbf{u} = 0 \quad \text{on } y = \epsilon \cos x.$$

Double expansion

Small Reynolds number, $Re \ll 1$. and small bump, $\epsilon \ll 1$.

$$
\begin{aligned}
u &\sim \overline{u}^0 + Re\overline{u}^i + \epsilon\left(\tilde{u}^0 + Re\tilde{u}^i\right) \\
v &\sim \quad + \epsilon\left(\tilde{v}^0 + Re\tilde{v}^i\right) \\
p &\sim \quad + \epsilon\left(\tilde{p}^0 + Re\tilde{p}^i\right)
\end{aligned}
$$

$\overline{u}^0$ Couette flow
$\overline{u}^i$ Inertial correction to Couette flow
$\tilde{u}^0$ Stokes flow over bump
$\tilde{u}^i$ Inertial correction to flow over bump

Couette flow and inertial correction

Couette flow for a flat bottom

$$\overline{u}^0 = \overline{U}^0(y)\cos t \quad \text{with} \quad \overline{U}^0 = Ay.$$

Couette flow and inertial correction

Couette flow for a flat bottom

$$\overline{u}^0 = \overline{U}^0(y) \cos t \quad \text{with} \quad \overline{U}^0 = Ay.$$

Inertial correction

$$\overline{u}^i = \overline{U}^i(y) \sin t \quad \text{with} \quad \overline{U}^i = \tfrac{1}{6}A(y - y^3).$$

Continues as base Couette flow reverses.

Wavy bottom perturbation of Stokes flow

$$\tilde{u}^0 = \tilde{U}^0(y)\cos x \cos t \quad \text{with} \quad \tilde{U}^0 = A(-1 + 4y - 3y^2).$$

This horizontal velocity is negative on crests, $x = 0$ and y small,

so sum with positive Couette flow vanishes (no slip) on crest.

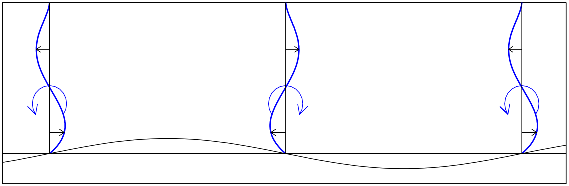
Inertial correction to wavy bottom disturbance

$$\tilde{u}^i = \tilde{U}^{i1}(y)\cos x \sin t + \tilde{U}^{i2}(y)\sin x \cos^2 t,$$

with

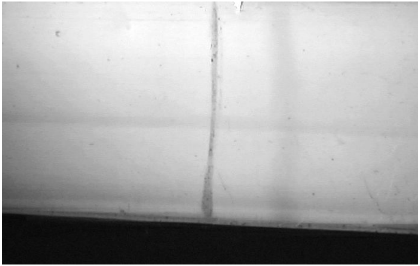
$$\begin{aligned}\tilde{U}^{i1} &= \tfrac{1}{60}A(-10 + 32y + 3y^2 - 40y^3 + 15y^4),\\ \tilde{U}^{i2} &= \tfrac{1}{60}A^2(-2y + 6y^2 - 10y^4 + 6y^5).\end{aligned}$$

Steady streaming part – from troughs to crests



Experimental check of steady streaming

To the right of a solid “ripple”.



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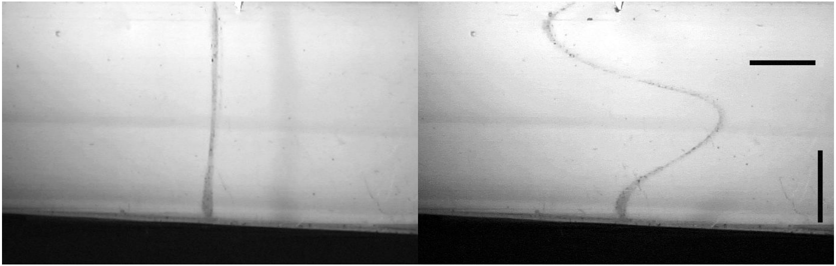

Initial dye filament, and after 39 complete oscillations

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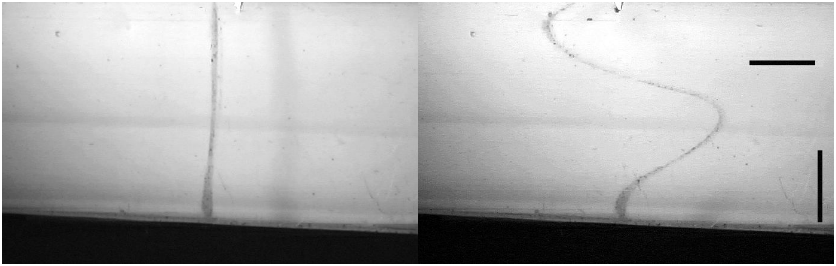

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WRONG direction!

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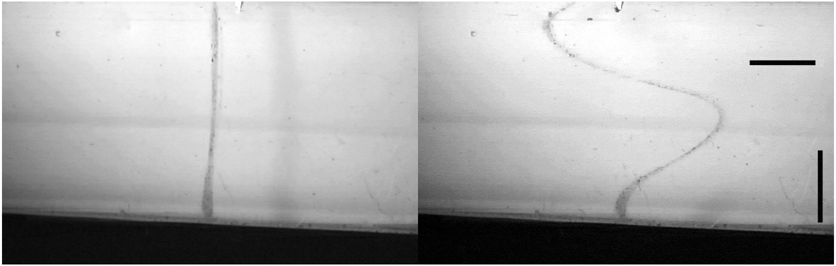

Initial dye filament, and after 39 complete oscillations

WRONG direction! But dye is Lagrangian. Different?

Calculation of Lagrangian mean flow

$$\mathbf{x}(t) \sim \mathbf{X}(T) + \delta\mathbf{x}(t),$$

First approximation: oscillate about $\mathbf{X}$

$$\dot{\delta\mathbf{x}} = \mathbf{u}(\mathbf{X}, t),$$

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First approximation: oscillate about $\mathbf{X}$

$$\dot{\delta\mathbf{x}} = \mathbf{u}(\mathbf{X}, t),$$

Second correction: mean drift

$$\mathbf{V}_{\mathrm{Stokes}} = \left\langle \delta\mathbf{x} \cdot \nabla\mathbf{u}|_{\mathbf{X}} \right\rangle,$$

Double expansion

Lagrangian mean flow needs inertia and wavy bottom:

$$V_{\mathrm{Stokes}} = \frac{1}{2}\epsilon Re\left[-\overline{U}^0\,\tilde{U}^{i1} + \overline{U}^i\,\tilde{U}^0 + \tilde{V}^0\frac{d\overline{U}^i}{dY} - \tilde{V}^{i1}\frac{d\overline{U}^0}{dY}\right]\sin X.$$

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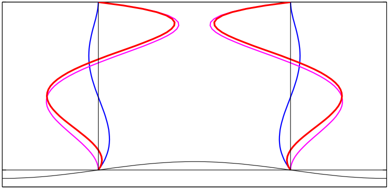
Hence

$$V_{\text{Stokes}} = \frac{1}{60}\epsilon ReA^2(6Y^2 - 2Y^3 - 25Y^4 + 21Y^5)\sin X.$$



Lagrangian mean flow larger than Eulerian and in opposite direction,

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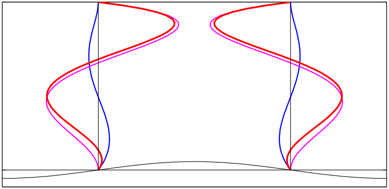

Lagrangian mean flow larger than Eulerian and in opposite direction, except near bottom.

Origin of Stokes drift

- Oscillation greater near crest, smaller near trough
- Inertial overshoot when top plate stops

Particle moving near top plate:

Origin of Stokes drift

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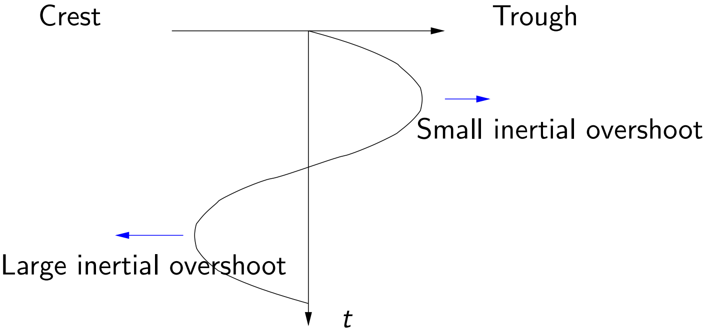

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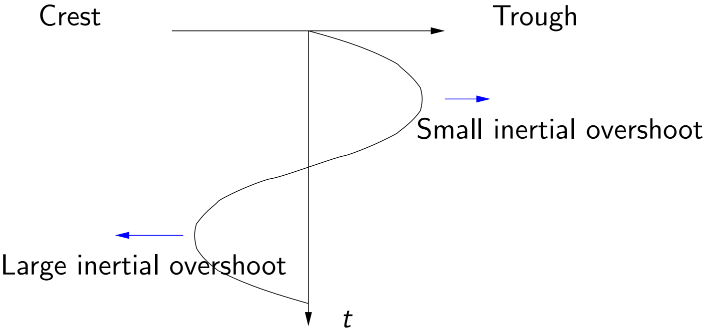
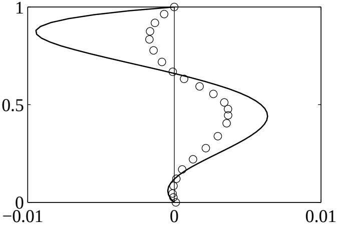


- Hence net drift towards crest near top

(Near bottom complicating effects of vertical flow)

Experimental check of Lagrangian drift



Poor agreement.

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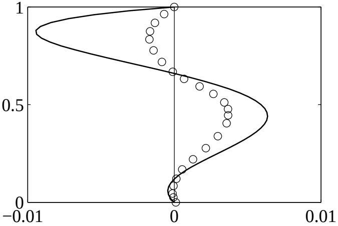

Poor agreement.

But large amplitude $A = 3.2$: top moves more than wavelength of ripple.

Stokes drift at $A = O(1)$ amplitudes

Double expansion again

$$
\begin{aligned}
x(t) &\sim \overline{x}^0 + Re\,\overline{x}^i + \epsilon\tilde{x}^0 + \epsilon Re\,\tilde{x}^i,\\
y(t) &\sim \overline{y}^0 \qquad\qquad + \epsilon\tilde{y}^0 + \epsilon Re\,\tilde{y}^i,
\end{aligned}
$$

with large oscillation with the base Couette flow

$$
\begin{aligned}
\overline{x}^0 &= X(T) + \overline{U}^0(Y)\sin t,\\
\overline{y}^0 &= Y(T).
\end{aligned}
$$

Example

One term in wavy-bottom correction to the Stokes flow

$$\dot{\tilde{y}}^0 = \tilde{V}^0(Y) \sin\left[X + \overline{U}^0(Y) \sin t\right] \cos t,$$

with displacements,

Example

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$$\dot{\tilde{y}}^0 = \tilde{V}^0(Y) \sin \left[X + \overline{U}^0(Y) \sin t \right] \cos t,$$

with displacements,
which can be integrated to

$$\tilde{y}^0 = -\tilde{V}^0(Y) \frac{\cos \left[X + \overline{U}^0(Y) \sin t \right]}{\overline{U}^0(Y)}.$$

Inertial correction to wavy-bottom flow

Problem for drift

$$\dot{\tilde{x}}^i = \tilde{u}^i + \tilde{y}^i \frac{\partial \overline{u}^0}{\partial y} + \tilde{y}^0 \frac{\partial \overline{u}^i}{\partial y} + \overline{x}^i \frac{\partial \tilde{u}^0}{\partial x},$$

$$\dot{\tilde{y}}^i = \tilde{v}^i + \overline{x}^i \frac{\partial \tilde{v}^0}{\partial x}.$$

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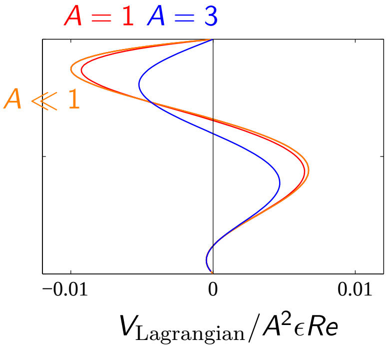
$$\dot{\tilde{y}}^i = \tilde{v}^i \qquad\qquad + \overline{x}^i \frac{\partial \tilde{v}^0}{\partial x}.$$

'Solution'

$$V_{\mathrm{Lagrangian}} = \epsilon Re \Bigg[\tilde{U}^{i1} J_0' + \tilde{U}^{i2} (J_0 + J_0'') + \frac{d\overline{U}^0}{dY} \left(\tilde{V}^{i1} J_0'' - (\tilde{V}^{i2} - \overline{U}^i \tilde{V}^0)(J_0' + J_0''') \right) - \frac{\tilde{V}^0}{\overline{U}^0} \frac{d\overline{U}^i}{dY} J_0' + \overline{U}^i \tilde{U}^0 (J_0 + J_0'') \Bigg] \sin X,$$

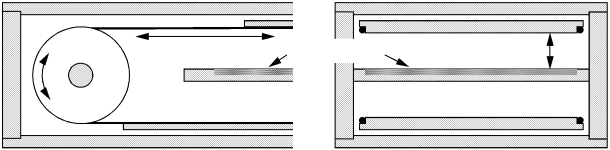
where $\langle \cos(z \sin\theta) \rangle = J_0(z)$ etc.

Result at large amplitude



Reduced effect due to averaging over large excursion

Second experiment in straight channel



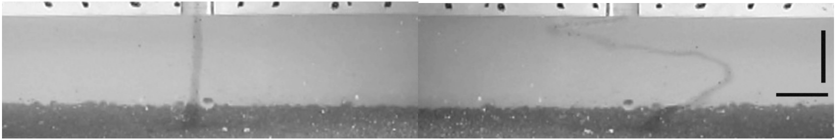
$$L = 2\,\mathrm{m},\ h = 11\ \&\ 16\,\mathrm{mm},\ U_0/\omega \le 200\,\mathrm{mm}.$$

Note balanced top and bottom channels, so no pressure gradient.

But particles slowly fall off ends of central plate.

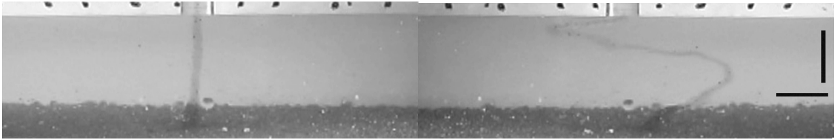
Second experimental check of Lagrangian mean flow

Initial dyed filament on erodible bed, and after 8 oscillations

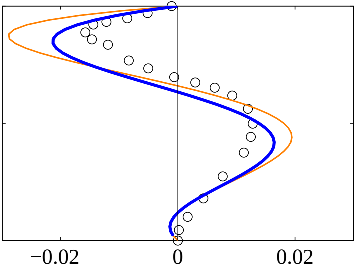


Second experimental check of Lagrangian mean flow

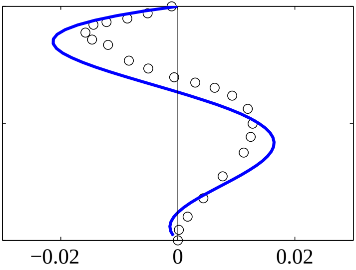
Initial dyed filament on erodible bed, and after 8 oscillations



Corrected theory $A = 2.0$, original $A \ll 1$ theory



Conclusions



- Dyed filament follows Lagrangian mean flow
- Shear at bed is Eulerian mean, from troughs to crests

Steady streaming exits, in correct direction for growth of dunes

References

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- F. Charru & E.J. Hinch (2005) *Ripple formation on a particle bed sheared by a viscous liquid. Part One: steady flow.* J. Fluid Mech. **550**, 111–121.
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