Drop formation of a power-law fluid on a thin film coating a vertical fibre

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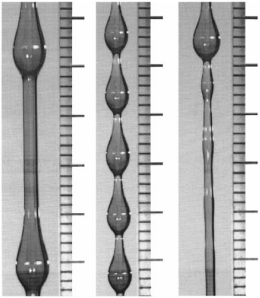
March 6, 2012

Motivation

Manufacture of polymeric and optical fibres.
The coating fluid is often non-Newtonian.

Shear-thinning
Duprat, Ruyer-Quil & Giorgiutti-Dauphiné

Phys. Fluids 2009



Newtonian

Kliakhandler, Davis & Bankoff JFM 2001

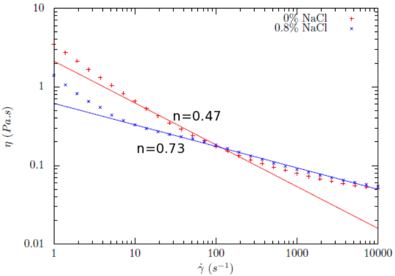
Governing equations

Constitutive equation

Power-law viscosity: $$\mu = \beta \left| \frac{\partial u}{\partial y} \right|^{n-1}$$

$$\sigma_{xy} = \beta \left| \frac{\partial u}{\partial y} \right|^{n} \text{sign}\left(\frac{\partial u}{\partial y} \right)$$

Xanthan solutions



Boulogne et al, Private Communication

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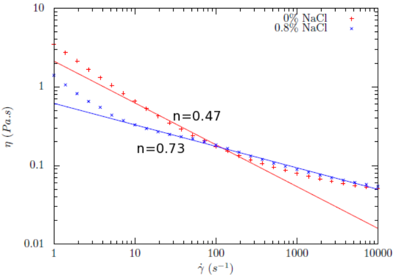

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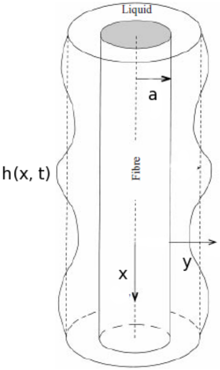
Geometry
Axisymmetric



Governing equations

Lubrication framework

Momentum: $$0 = -\frac{\mathrm{d}p}{\mathrm{d}x} + \rho g + \frac{\partial \sigma_{xy}}{\partial y}$$

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Volume flux: $$Q = \beta^{-\frac{1}{n}} \frac{n}{2n+1} \left(\rho g - \frac{\mathrm{d}p}{\mathrm{d}x} \right)^{\frac{1}{n}} h^{(2+\frac{1}{n})}$$

Note:$(\cdot)^{\frac{1}{n}} = \mathrm{sign}(\cdot)|\cdot|^{\frac{1}{n}}$

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Mass conservation: $$h_t + Q_x = 0$$

Governing equations

Non-dimensionalisation

Lengthscales:

- Fibre radius, a, in x direction.
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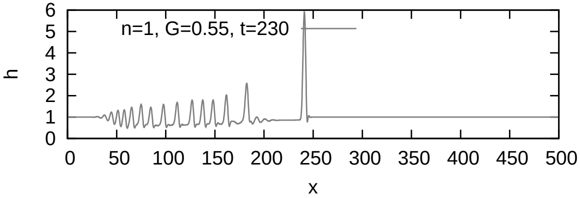
$$\boxed{h_t + \left(h^{2+\frac{1}{n}}(G + (h + h_{xx})_x)^{\frac{1}{n}}\right)_x = 0}$$

where Bond number $G = \frac{\rho g a^3}{\gamma h_0}$.

G small (thicker film):



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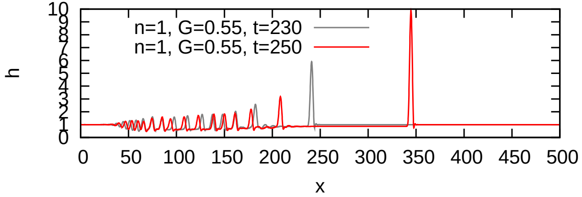

Ever growing large pulses

Time-dependent numerical simulations

Periodic forcing at inlet: $\omega = 1$

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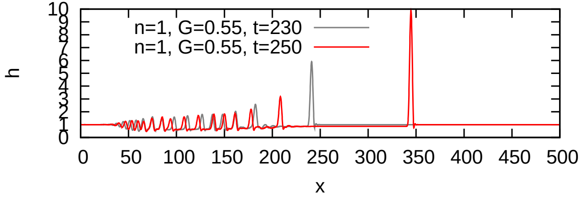

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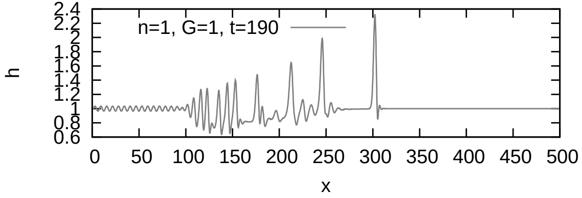

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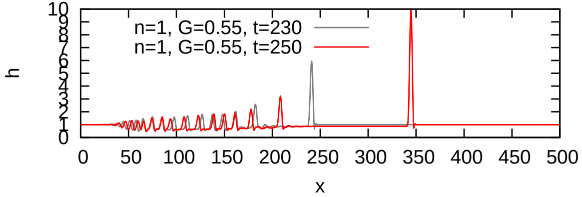

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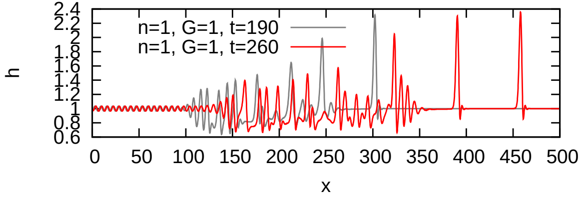

Solitary waves of similar amplitude and speed

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Period in x

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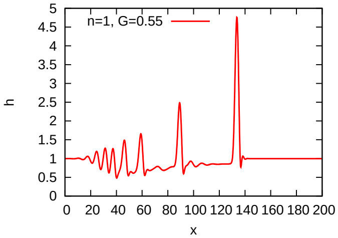

- Ever growing large pulses.
- Coalescence cascade.

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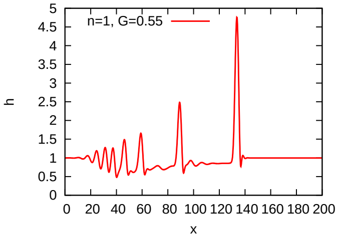

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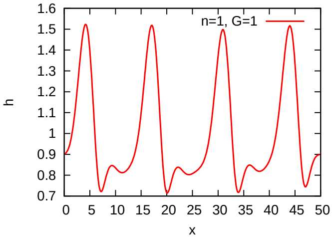

- Train of regular pulses of similar amplitude and speed.
- Weakly interacting with each other, but don't coalesce.

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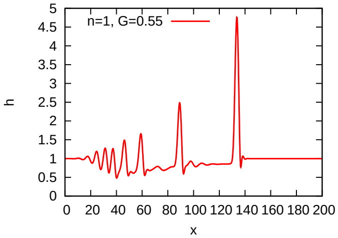

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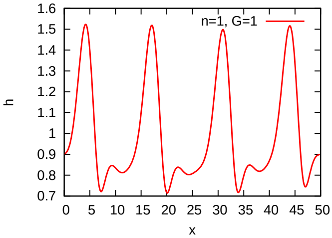

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This talk: Equilibrium pulses? When? Properties?

In the frame of the solitary waves traveling with speed c:

$$(G + (h + h_{xx})_x) = \frac{\left(c(h-1) + G^{\frac{1}{n}}\right)^n}{h^{2n+1}}$$

$$h \to 1, \quad \text{as} \quad x \to \pm\infty$$

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Numerically construct the stationary solitary waves.

- Integrating from $x = -\infty$ and from $x = +\infty$ to $x = 0$.
- Hence need to find starting conditions at $x = \pm\infty$.

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Stationary solitary waves

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Three solutions of exponential form:

- $\tilde{h}_1 = a_1 \mathrm{e}^{m_1 x}$
  m_1 real and positive.
- $\tilde{h}_{2,3} = a_{2,3} \mathrm{e}^{m_{2,3} x}$
  $m_{2,3}$ complex conjugates with negative real parts.

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- $\tilde{h}_1 = a_1 e^{m_1 x}$
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  Use in 'Back' (1 DoF).
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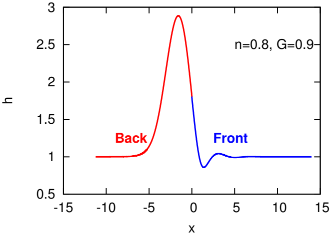
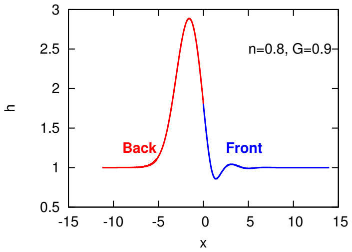
  Use in 'Front' (2 DoF).



For fixed G:

1. Shoot from Back, with $a_2 = a_3 = 0$. Stop when $h'' = 0$, $h' < 0$.



- Monotonic increase of h at Back.

Stationary solitary waves

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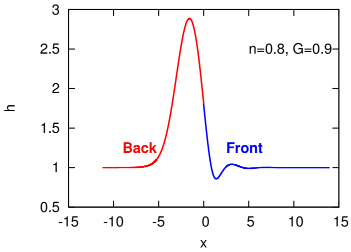

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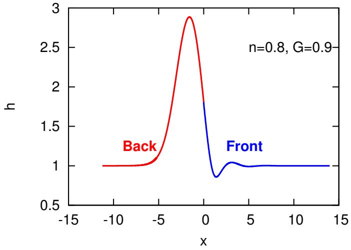

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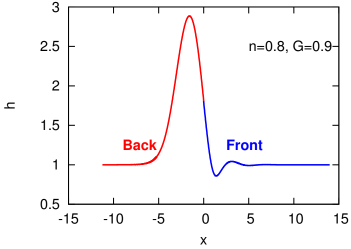
4. Vary speed c to match h'.



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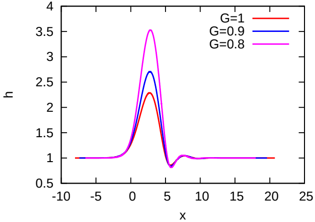
Stationary solitary waves

Results: $n = 1$ Kalliadasis & Chang, J. Fluid Mech. 1994



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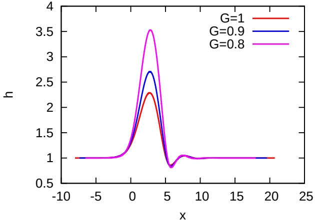

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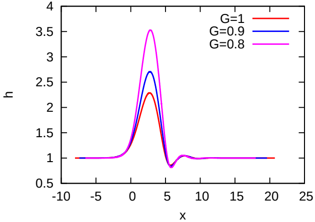

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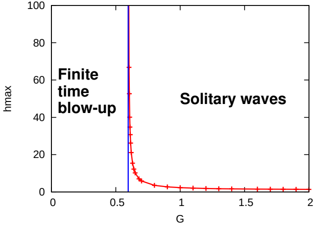

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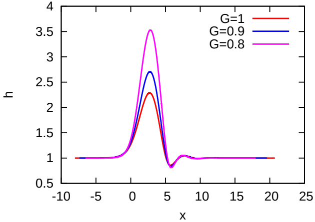

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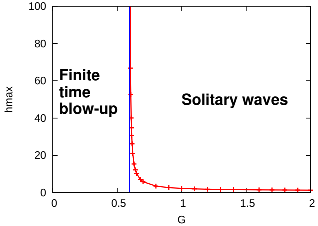

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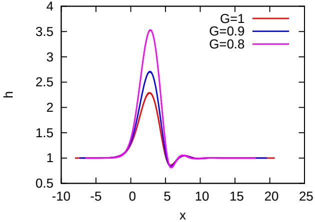

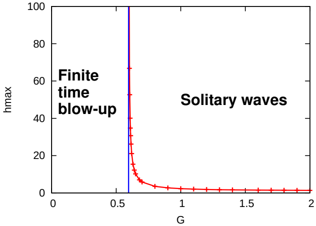

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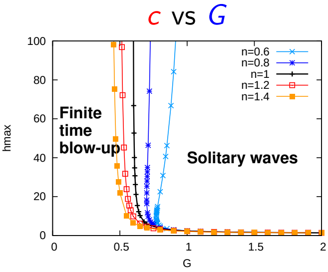
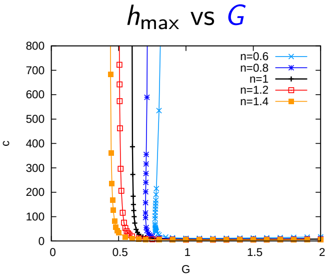
Agreement with experiment Quéré, Europhys. Lett. 1990:

- Critical h_c to observe disturbance $\propto a^3$.
- $G = \frac{\rho g a^3}{\sigma h_0} \Rightarrow h_c \propto a^3$ at $G = G_0$.

Stationary solitary waves

Results: various n





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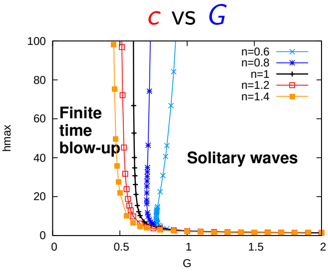

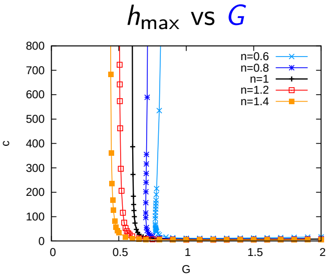

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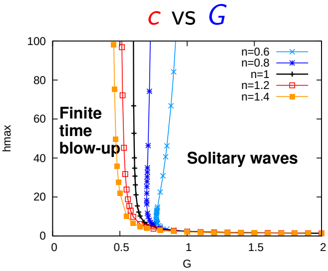

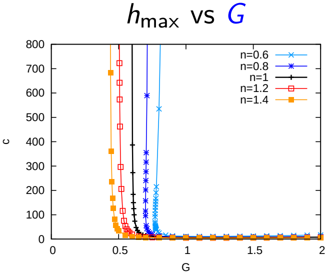

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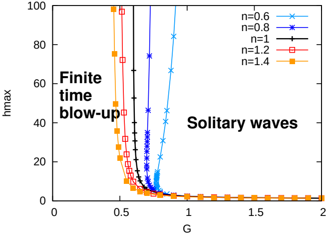
What determines critical G_0? Relationship of h and c with G?

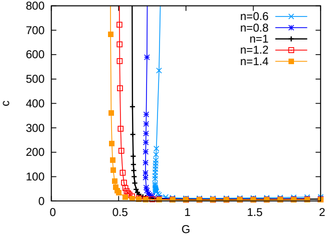
Stationary solitary waves

Results: various n

c vs G



$h_{\max}$ vs G



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What determines critical G_0? Relationship of h and c with G?

Look at large fast stationary solitary waves close to G_0.

Large fast solitary waves

Pulse divided into 3 regions:

Large fast solitary waves

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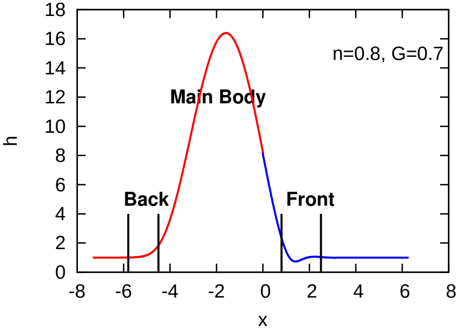
- 'Main body' region: h big, $x \sim O(1)$.



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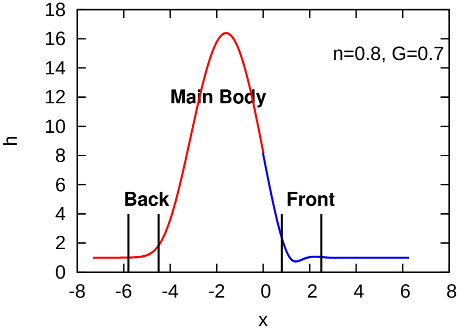

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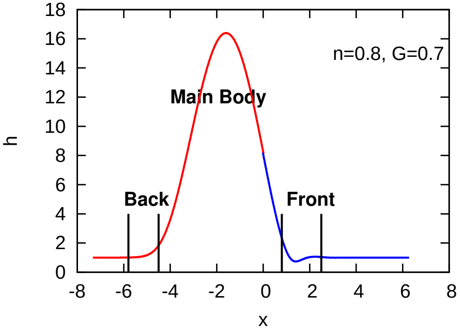

Asymptotic analysis for each region, and match.

h big, $x \sim O(1)$

$$(G + (\mathbf{h} + \mathbf{h_{xx}})_{\mathbf{x}}) = \frac{\left(c(h-1) + G^{\frac{1}{n}}\right)^n}{h^{2n+1}}$$

Large fast solitary waves

Main body region: leading order

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Solution: constant capillary pressure ($p = \frac{1}{2} h_{\text{max}}$)

$$h = \frac{1}{2} h_{\mathrm{max}} (1 - \cos x) \quad \text{in} \quad 0 \leq x \leq 2\pi.$$

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For matching,

$$h \sim \frac{1}{4} h_{\max}(x - x_0)^2,$$

with $x_0 = 0$ at the Back and $x_0 = 2\pi$ at the Front.

$h \sim O(1)$, x small

$$(G + (h + \mathbf{h_{xx}})_{\mathbf{x}}) = \frac{\left(c(\mathbf{h} - \mathbf{1}) + G^{\frac{1}{n}}\right)^{n}}{\mathbf{h^{2n+1}}}$$

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Large fast solitary waves

Transition regions: leading order

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Solutions towards 'Main Body'

$$h \sim \tfrac{1}{2} P_\pm \xi^2 + Q\xi + R_\pm \quad \text{as} \quad \xi \to \pm\infty$$

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Use 1 DoF to redefine origin so $Q = 0$.

Large fast solitary waves

Matching: leading order

At Back $1 - 1(Q = 0) = 0$ DoF: P_+ and R_+ uniquely determined.

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Main body: $h \sim \frac{1}{4} h_{\text{max}} (x - x_0)^2 \quad \text{near} \quad x_0 = 0,\, 2\pi$.

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Transition regions: $h \sim \frac{1}{2} P_\pm \xi^2 + R_\pm$ as $\xi \to \pm\infty$.
Matching:

$$P_- = P_+$$

So now P_- unique and hence R_-.

Large fast solitary waves

Matching: leading order

At Back $1 - 1(Q = 0) = 0$ DoF: P_+ and R_+ uniquely determined.
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Matching:

$$P_- = P_+$$

So now P_- unique and hence R_-.

$$\frac{1}{2} P(\xi = c^{n/3}(x - x_0))^2 = \frac{1}{4} h_{\text{max}}(x - x_0)^2$$

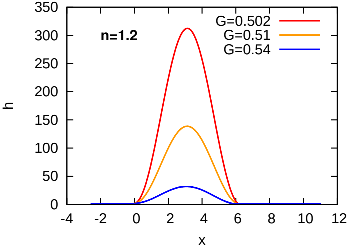
$$\boxed{h_{\text{max}} = 2Pc^{2n/3}}$$

Note: capillary pressure in the main body $p = \frac{1}{2} h_{max} = Pc^{2n/3}$.

Large fast solitary waves

Checking scalings

Original profile:



Large fast solitary waves

Checking scalings

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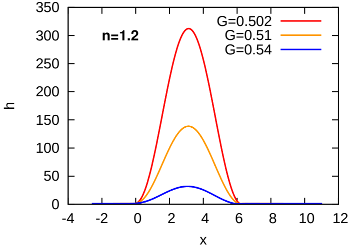
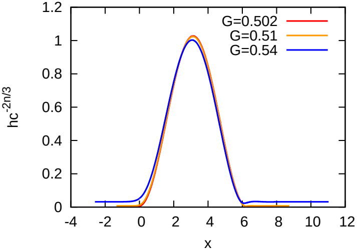


Main body:
height scaled by $c^{-2n/3}$



Large fast solitary waves

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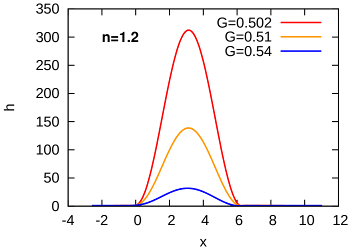

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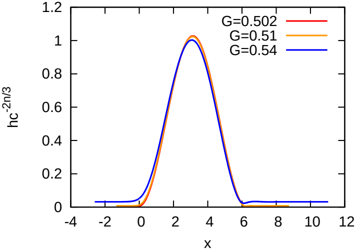
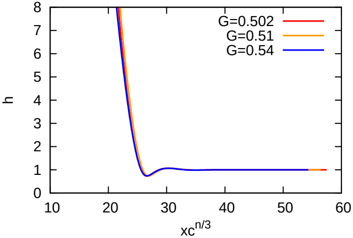


'Front' transition region:
width scaled by $c^{n/3}$



So far have $h_{\max}(c)$. G yet to appear

Large fast solitary waves

Critical G

So far have $h_{\max}(c)$. G yet to appear

Transition regions: $h \sim \frac{1}{2}P\xi^2 + R_{\pm}$.

- Different apparent film thickness, $R_{\pm}$, at 'Back' and 'Front'.

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Need 1st correction of Main Body: $h \sim c^{2n/3}h_0 + h_2$

$$(\mathbf{G} + (\mathbf{h} + \mathbf{h_{xx}})_\mathbf{x}) = \frac{\left(c(h-1) + G^{\frac{1}{n}}\right)^n}{h^{2n+1}}$$

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$$h_2 = -G_0(x - \sin x) + R_+ \quad \text{in} \quad 0 \le x \le 2\pi.$$

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Matching gives critical G_0:

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Finding $R_\pm$ accurately

Modified Bretherton equation

$$h''' = \frac{(h-1)^n}{h^{2n+1}}$$

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$$\tilde{h} = A(\xi - \xi_0)^{\frac{3}{1-n}}, \quad n < 1$$

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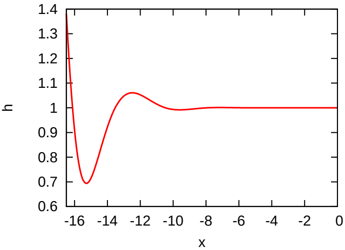
Solution at 'Front' ($n = 0.8$):



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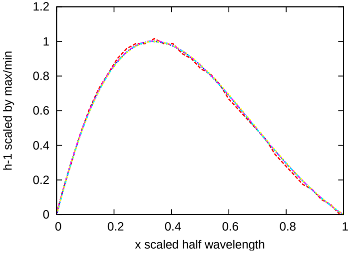

Finding $R_\pm$ accurately

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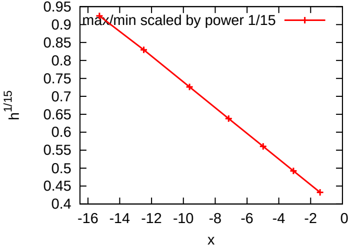

Finding $R_\pm$

Numerics

$$h''' = \frac{(h-1)^n}{h^{2n+1}}$$

$$h \sim \frac{1}{2}P\xi^2 + R_\pm + S(n)\xi^{1-2n} + T(n)\xi^{-1-2n} + \dots$$

with $S(n) = \frac{2^{n+1}}{(1-2n)(-2n)(-1-2n)P^{n+1}}$, $T(n) = \frac{2^{n+2}((n+1)R_\pm + n)}{P^{n+2}(-1-2n)(-2-2n)(-3-2n)}$.

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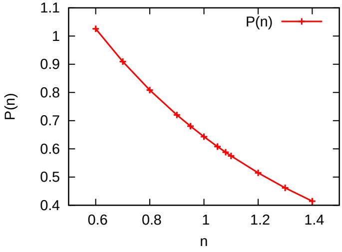
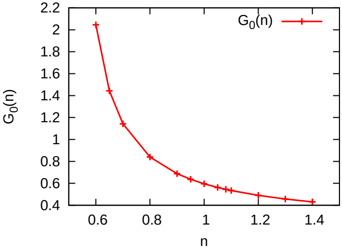
Least-square-fit[100:150]: $P(n) \pm 0.00001$, $R_\pm(n) \pm 0.001$.

$P(n)$



$G_0(n) = (R_+ - R_-)/2\pi$



Large fast solitary waves

c as a function of G

So far have $h_{\max}(c)$ and critical G_0. Yet to find $c(G)$.

Large fast solitary waves

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$$\left(G + (\mathbf{h} + \mathbf{h_{xx}})_\mathbf{x}\right) = \frac{\left(\mathbf{c}(\mathbf{h}-1) + G^{\frac{1}{n}}\right)^{\mathbf{n}}}{\mathbf{h}^{\mathbf{2n+1}}}$$

Need 2nd correction: $\quad h \sim c^{2n/3} h_0 + h_2 + c^{-(2n-1)n/3} h_3$

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$$h_3 \sim k_1 (x - x_0)^{1-2n} + D_\pm$$

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$$(h_3 + h_{3xx})_x = \left(\frac{1}{P^{n+1}(1-\cos x)^{n+1}} - G_1\right)$$

Near $x = x_0$

$$h_3 \sim k_1 (x - x_0)^{1-2n} + D_\pm$$

- The singular k_1 term matches $S(n)$ in transition regions.
- $D_\pm$ different at the 'Back' and 'Front'.
- No terms to match with them from transition regions.
- Need an expansion from G:

$$\boxed{G = G_0 + c^{(-(2n-1)n/3} G_1}$$

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Large fast solitary waves

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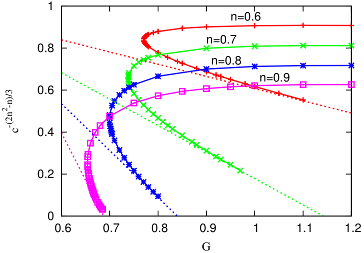
$$G_1 = (D_- - D_+)/2\pi$$

$$\boxed{G = G_0 + c^{-(2n-1)n/3} G_1}$$

G_1 determined numerically by finding the $D_\pm$.

$$\boxed{G = G_0 + c^{-(2n-1)n/3} G_1}$$

$n < 1$:



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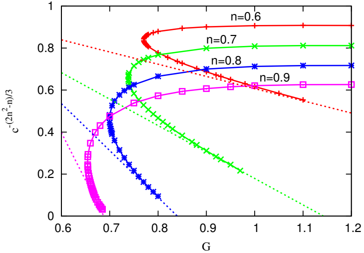

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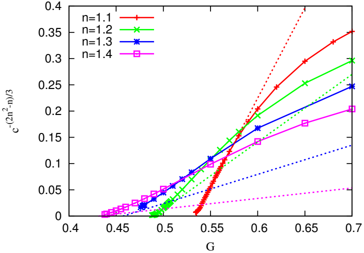
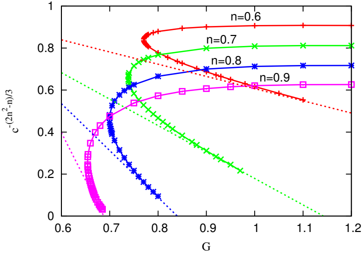


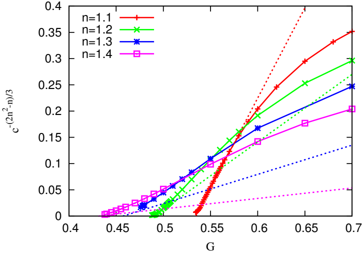
- When $n < 1$, $G_1 < 0$. Negative slope at G_0.
- When $n > 1$, $G_1 > 0$. Positive slope at G_0.

$$\boxed{G = G_0 + c^{-(2n-1)n/3} G_1}$$

$n < 1$:



$n > 1$:



- When $n < 1$, $G_1 < 0$. Negative slope at G_0.
- When $n > 1$, $G_1 > 0$. Positive slope at G_0.
- When $n = 1$, $G_1 = 0$. No relationship between G and c yet.

More terms

transition regions

With scaling $\xi = c^{n/3}(x - x_0)$,

$$h_{\xi\xi\xi} = \frac{(h-1)^n}{h^{2n+1}} - c^{-2n/3} h_\xi - c^{-n} G + c^{-1} \frac{n(h-1)^{n-1} G^{1/n}}{h^{2n+1}} + \dots$$

More terms

transition regions

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Expand h as

$$h \sim h_0 + c^{-2n/3} h_2 + c^{-n} h_3 + c^{-1} h_4 + c^{-4n/3} h_5 + \dots$$

More terms

transition regions

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$$h_0''' = \frac{(h-1)^n}{h^{2n+1}}, \qquad h_0 \sim \frac{P}{2}\xi^2 + \mathbf{R}_\pm + k_1 x^{1-2n}$$

$$h_2''' = \frac{(h_0-1)^{n-1}\left(-(n+1)h_0 + (2n+1)\right)}{h_0^{2n+2}} h_2 - \mathbf{h_0'}, \qquad h_2 \sim -\frac{P}{4!}\xi^4 + \frac{a_{2\pm}}{2}\xi^2 + \mathbf{c_{2\pm}} + k_2 \xi^{3-2n}$$

$$h_3''' = \frac{(h_0-1)^{n-1}(-(n+1)h_0 + (2n+1))}{h_0^{2n+2}} h_3 - \mathbf{G_0}, \qquad h_3 \sim -\frac{G_0}{3!}\xi^3 + \frac{a_{3\pm}}{2}\xi^2 + \mathbf{c_{3\pm}}$$

$$h_4''' = \frac{(h_0-1)^{n-1}(-(n+1)h_0 + (2n+1))}{h_0^{2n+2}} h_4 + \frac{n(h_0-1)^{n-1} G_0^{1/n}}{h_0^{2n+1}}, \qquad h_4 \sim \frac{1}{2} a_{4\pm}\xi^2 + c_{4\pm}$$

More terms

main body

With $h = c^{2n/3} H$,

$$(H + H_{xx})_x = -c^{-2n/3} G + c^{-(2n+1)n/3} \frac{\left(1 - \frac{c^{-2n/3}}{H} + \frac{G^{1/n}(c^{-1-2n/3})}{H}\right)^n}{H^{n+1}}.$$

More terms

main body

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Expand H as

$$H \sim H_0 + c^{-2n/3} H_2 + c^{-(2n+1)n/3} H_3 + c^{-n} H_4 + c^{-1} H_5 + c^{-4n/3} H_6 + \dots$$

and G as

$$G \sim G_0 + G_1 c^{-(2n-1)n/3} + G_2 c^{-2n/3} + \dots$$

More terms

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and G as

$$G \sim G_0 + G_1 c^{-(2n-1)n/3} + G_2 c^{-2n/3} + \dots$$

$$\begin{aligned}
H_0' + H_0''' &= 0, & H_0 &= P(1 - \cos x) \\
H_2' + H_2''' &= -G_0 & H_2 &= G_0 (\sin x - x) + A_2 + C_2 \cos x \\
H_3' + H_3''' &= -G_1 + \frac{1}{P^{n+1}(1 - \cos x)^{n+1}}, & H_3 &= k_1 x^{1-2n} + D_{\pm} - G_1 x + k_2 x^{3-2n} \\
H_4' + H_4''' &= 0, & H_4 &= A_4 + B_4 \sin x + C_4 \cos x \\
H_5' + H_5''' &= 0, & H_5 &= A_5 + B_5 \sin x + C_5 \cos x \\
H_6' + H_6''' &= -G_2, & H_6 &= G_2(\sin x - x) + A_6 + B_6 \sin x + C_6 \cos x
\end{aligned}$$

More terms

Matching: transition regions

Transition regions=

$$
\begin{array}{rllllll}
 & h_0 & h_2 & h_3 & h_4 & h_5 & \\
c^{\frac{2n}{3}}\Big[& \frac{P}{2}x^2 & -\frac{P}{4!}x^4 & & & +\frac{P}{6!}x^6 & +\dots\Big] \\
+c^0\Big[& \color{red}{R_\pm} & +\frac{a_2}{2}x^2 & -\frac{G_0}{3!}x^3 & & -\frac{a_2}{4!}x^4 & +\dots\Big] \\
+c^{-\frac{2n^2}{3}+\frac{n}{3}}\Big[& k_1x^{1-2n} & +k_2x^{3-2n} & & & +k_3x^{5-2n} & +\dots\Big] \\
+c^{-\frac{n}{3}}\Big[& & & +\frac{a_3}{2}x^2 & & & +\dots\Big] \\
+c^{\frac{2n}{3}-1}\Big[& & & & \frac{a_4}{2}x^2 & & +\dots\Big] \\
+c^{-\frac{2n}{3}}\Big[& & \color{red}{c_{2\pm}} & & & +\frac{a_5}{2}x^2 & +\dots\Big]
\end{array}
$$

More terms

Matching: main body region

Main body=

$$
\begin{aligned}
& c^{\frac{2n}{3}} && \left[\tfrac{P}{2}x^2 - \tfrac{P}{4!}x^4 + \tfrac{P}{6!}x^6 + \dots\right] \\
& +c^{0} && \left[-G_0 x_0 + A_2 + C_2 - \tfrac{C_2}{2}x^2 - \tfrac{G_0}{3!}x^3 + \tfrac{C_2}{4!}x^4 + \dots\right] \\
& +c^{-\frac{2n^2}{3}+\frac{n}{3}} && \left[k_1 x^{1-2n} - G_1 x_0 + D_\pm + k_2 x^{3-2n} + k_3 x^{5-2n} + \dots\right] \\
& +c^{-\frac{n}{3}} && \left[A_4 + C_4 - \tfrac{C_4}{2}x^2 + \dots\right] \\
& +c^{\frac{2n}{3}-1} && \left[A_5 + C_5 - \tfrac{C_5}{2}x^2\right] \\
& +c^{-\frac{2n}{3}} && \left[-G_2 x_0 + A_6 + C_6 - \tfrac{C_6}{2}x^2 - \tfrac{G_2+B_6}{3!}x^3 + \dots\right]
\end{aligned}
$$

More terms: Results

At c^0:

$$G_0 = (R_+ - R_-)/2\pi$$

At $c^{-(2n^2-n)/3}$:

$$G_1 = -(D_+ - D_-)/2\pi$$

At c^{-1}:

$$G_2 = (c_{2+} - c_{2-})/2\pi$$

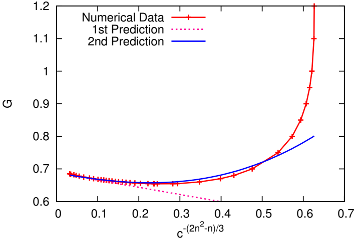
Hence,

$$\boxed{G = G_0 + G_1 c^{-(2n-1)n/3} + G_2 c^{-2n/3}}$$

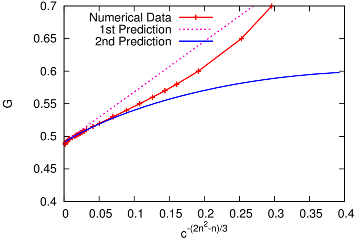
More terms: Results

$$\boxed{G = G_0 + G_1 c^{-(2n-1)n/3} + G_2 c^{-2n/3}}$$

$n = 0.9 \quad G$ vs $c^{-(2n-1)n/3}$



$n = 1.2 \quad G$ vs $c^{-(2n-1)n/3}$



More terms: Results

$$\boxed{G = G_0 + G_1 c^{-(2n^2-n)/3} + G_2 c^{-2n/3}}$$

When $n = 1$, $G_1 = 0$, so

$$G = G_0 + G_2 c^{-2/3}$$

Need even more terms!

Transition regions=

$$
\begin{array}{rlllll}
 & h_0 & h_2 & h_3 & h_4 & \\
c^{2/3}\Big[& \frac{P}{2}x^2 & -\frac{P}{4!}x^4 & & +\frac{P}{6!}x^6 & +\dots\Big] \\
+c^0\Big[& R_\pm & +\frac{a_2}{2}x^2 & -\frac{G_0}{3!}x^3 & -\frac{a_2}{4!}x^4 & +\dots\Big] \\
+c^{-1/3}\Big[& -\frac{2}{3P^2x} & & +\frac{a_3}{2}x^2 & +\frac{11}{1080P^2}x^3 & +\dots\Big] \\
+c^{-2/3}\Big[& & +c_{2\pm} & & +\frac{a_4}{2}x^2 & +\dots\Big] \\
+c^{-1}\log c\Big[& & & +\frac{4G_0}{9P^3} & & +\dots\Big] \\
+c^{-1}\Big[& \frac{2(1+2R_\pm)}{15P^3x^3} & +\frac{8R_\pm+4+20a_2}{15P^3x} & +\frac{4G_0}{3P^3}\log x + c_{3\pm} & & +\dots\Big]
\end{array}
$$

$n = 1$ Newtonian fluid, even more terms

Matching: main body region

Main body =

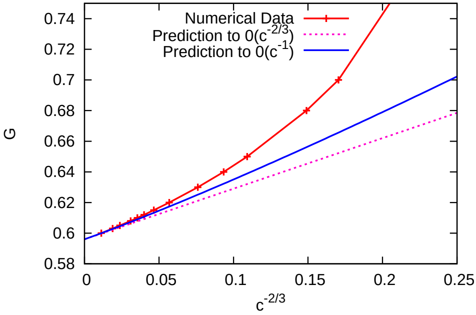
$$
\begin{aligned}
& c^{2/3} && \left[\tfrac{P}{2}x^2 - \tfrac{P}{4!}x^4 + \tfrac{P}{6!}x^6 + \dots\right] \\
& +c^{0} && \left[-G_0 x_0 + A_2 + C_2 - \tfrac{C_2}{2}x^2 - \tfrac{G_0}{3!}x^3 + \tfrac{C_2}{4!}x^4 + \dots\right] \\
& +c^{-1/3} && \left[-\tfrac{2}{3P^2 x} + (A_3 + C_3) + (\tfrac{1}{18P^2} + B_3)x - \tfrac{C_3}{2}x^2 + (\tfrac{1}{1080P^2} - \tfrac{B_3}{3!})x^3 \dots \right. \\
& +c^{-2/3} && \left[-G_2 x_0 + A_4 + C_4 + B_4 x - \tfrac{C_4}{2}x^2 - \tfrac{G_2}{3!}x^3 + \dots\right] \\
& +c^{-1}\log c && \left[A_5 + C_5 - \tfrac{C_5}{2}x^2 + \dots\right] \\
& +c^{-1} && \left[\tfrac{2(1+2R_\pm)}{15P^3x^3} + \tfrac{4(1+2A_2-3C_2)}{15P^3x} + \tfrac{4G_0}{3P^3}\log x - G_3 x_0 + A_6 + C_6 \dots\right]
\end{aligned}
$$

$n = 1$ Newtonian fluid, even more terms

Results

At c^0: $G_0 = (R_+ - R_-)/2\pi$
At $c^{-2/3}$: $G_2 = (c_{2+} - c_{2-})/2\pi$
At c^{-1}: $G_3 = (c_{3+} - c_{3-})/2\pi$
Hence,

$$\boxed{G = G_0 + G_2 c^{-2/3} + G_3 c^{-1}}$$



Future Work

- What happens at big G? $\surd$

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- Comparison with experimental data.

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Future Work

- What happens at big G? $\surd$
- Comparison with experimental data.
- Stability of the two branches for $n < 1$.
- Relax the thin film approximation?
- Normal stress effect.

The end

Thank you for your attention!

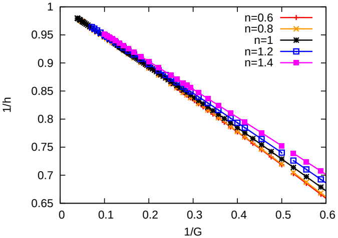
Big G

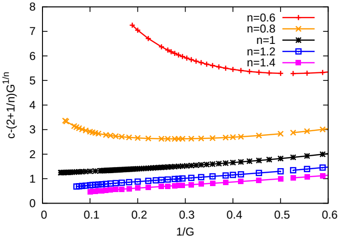
$$\left(G + (h + h_{xx})_x\right) = \frac{\left(c(h-1) + G^{\frac{1}{n}}\right)^n}{h^{1+2n}}$$

$$h \sim 1 + \frac{h_1}{G}$$

$$\Rightarrow G + \frac{h_{1x}}{G} + \frac{h_{1xxx}}{G} = G\left(1 + nG^{-1-\frac{1}{n}} c h_1\right)\left(1 - (1+2n)\frac{h_1}{G}\right)$$

$$\Rightarrow c \sim \left(2 + \frac{1}{n}\right) G^{\frac{1}{n}}$$





Not working yet for $n < 1$!

Big G

Expanding c to the next order suggests:

$$c \sim \left(2 + \frac{1}{n}\right) G^{\frac{1}{n}} + c_1 G^{\frac{1}{n}-1}$$