Fast Flow of Oldroyd-B fluid through a slowly varying contraction and constriction in a channel

John Hinch

CMS-DAMTP, University of Cambridge

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from talking with Evgeniy Boyko & Howard Stone

Work in progress

Oldroyd-B model fluid   simplest viscous + elastic

$$\boldsymbol{\sigma} \quad = \quad -p\mathsf{I} \quad + \quad 2\mu_0\mathsf{E} \quad + \quad G\mathsf{A}$$

stress   viscous   elastic

μ_0 viscosity G elastic modulus

with A micro-structure deforming with the flow

$$\frac{D\mathsf{A}}{Dt} = \quad \mathsf{A}\cdot\nabla\mathsf{u} + \nabla\mathsf{u}^T\cdot\mathsf{A} \quad -\frac{1}{\tau}(\mathsf{A}-\mathsf{I})$$

deform with flow   relaxes

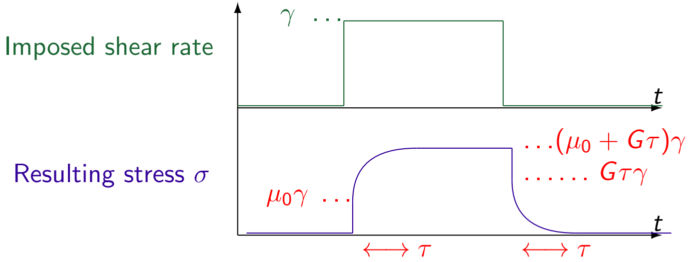
τ relaxation time

Simple model with three parameters –

bead-and-spring model, fits Boger fluids, used in numerics

Find behaviour.
Understand predicted behaviour.
Correct behaviour?

Transient behaviour – Linear visco-elasticity, common to all fluid models



- Early viscosity μ_0
- Takes τ to build up to steady state:

  steady deformation $=$ shear rate γ $\times$ memory time τ
- Steady state viscosity $\mu_0 + G\tau$

Fast through contraction $\rightarrow$ no time to build up elastic stress $\rightarrow$ less pressure drop?

(Stress relaxation is a special property of non-Newtonian fluids, which is not in elastic solids nor viscous liquids)

Normal Stresses – deforming with the flow 1

$$\frac{DA}{Dt} = \mathrm{A} \cdot \nabla \mathrm{u} + \nabla \mathrm{u}^T \cdot \mathrm{A}$$

Steady uniform simple shear $\mathrm{u} = (\gamma y, 0, 0)$,

Fibre | sheared to / with streamline component –.

	A_{yy}		$A_{yx} + A_{xy}$		A_{xx}
	$\|\|$	sheared to	$\|- + -\|$	sheared to	$2--$
Deformation	1		$\gamma\tau$		$2(\gamma\tau)^2$
	$\sigma_{yy} = -p$		$\sigma_{yx} = \sigma_{xy} = (\mu_0 + G\tau)\gamma$		$\sigma_{xx} = -p + 2G(\gamma\tau)^2$

Normal stress $2G(\gamma\tau)^2$ = tension in streamlines

Weissenberg and Deborah numbers

$$\sigma_{yy} = -p, \quad \sigma_{yx} = \sigma_{xy} = (\mu_0 + G\tau)\gamma, \quad \sigma_{xx} = -p + 2G(\gamma\tau)^2$$

Lubrication normally dominated by shear, σ_{xy}.

Can promote σ_{xx} with very high shear-rates if $\gamma\tau \sim\ L/h$ (gap $h \ll$ length L). Need:

$$\text{Weissenberg} \quad Wi = \gamma\tau = \frac{U\tau}{h} = O(\frac{L}{h}) \gg 1, \qquad \text{i.e.} \quad \text{Deborah} \quad De = \frac{U\tau}{L} = O(1).$$

$De = O(1)$: relaxation time $\tau \sim$ residence time in contraction L/U.

Scalings

For contraction over length L, with height $h(x)$, and volume flux Q, scale

$$x \text{ by } L, \quad y \text{ by } h_0 = h(x=0), \quad h(x) \text{ by } h_0,$$
$$u \text{ by } Q/h_0, \quad v \text{ by } Q/L, \quad t \text{ by } h_0 L/Q, \quad p \text{ by } \mu Q/h_0^3,$$
$$A_{yy} \text{ by } 1, \quad A_{xy} \text{ by } L/h_0, \quad A_{xx} \text{ by } (L/h_0)^2.$$

Parameters Deborah $De = Q\tau/h_0 L$, elasticity $c = G\tau/\mu_0$, contraction ratio $h_m = h(L)/h(0)$.

Transform (x, y) to (x, η) with $\eta = \dfrac{y}{h(x)}$ so contraction is $0 \le x \le 1 \quad 0 \le \eta \le 1$.

(tweak orthogonal curvilinear coordinates).

Governing equations

Mass

$$\frac{\partial (hu)}{\partial x} + \frac{\partial v}{\partial \eta} = 0.$$

Momentum

$$0 = -\frac{dP}{dx} + \frac{1}{h^2}\frac{\partial^2 u}{\partial \eta^2} + \frac{c}{De}\left(\frac{1}{h}\frac{\partial (hA_{11})}{\partial x} + \frac{1}{h}\frac{\partial A_{12}}{\partial \eta}\right),$$

Constitutive equation

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + \frac{v}{h}\frac{\partial}{\partial \eta}\right) A_{11} - 2eA_{11} \; -2\gamma_1 A_{12} = -\frac{1}{De}A_{11},$$
$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + \frac{v}{h}\frac{\partial}{\partial \eta}\right) A_{12} - \gamma_2 A_{11} \quad -\gamma_1 A_{22} = -\frac{1}{De}A_{12},$$
$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + \frac{v}{h}\frac{\partial}{\partial \eta}\right) A_{22} - 2\gamma_2 A_{12} \; +2eA_{22} = -\frac{1}{De}(A_{22} - 1),$$

where

$$\gamma_1 = \frac{1}{h}\frac{\partial u}{\partial \eta},$$
$$\gamma_2 = h\frac{\partial}{\partial X}\left(\frac{v}{h}\right),$$
$$e = \frac{\partial u}{\partial X}.$$

Numerical approach

Given the elastic stresses, find the flow:

- Integrate momentum from centre line $\to$ shear-rate $\partial u/\partial\eta$,
- Integrate shear-rate from wall $\to$ flow (u, v),
- Integrate flow $\to$ flux Q,
- Adjust pressure gradient to make $Q = \mathrm{const}$.

Find

$$\frac{dp}{dx} = -\frac{3Q}{h^3} + 3\int_0^1 \tfrac{1}{2}(1-\eta^2)E(\eta)\,d\eta,$$

where E is the divergence of the elastic stress.

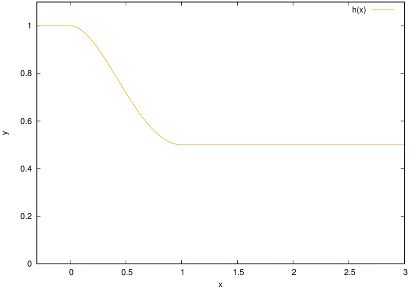
Use flow to time-step elastic stresses to steady state.

Finite differences, second-order accurate. No Poisson problem, so fast.

..numerical approach

Smooth Contraction:

$$h(x) = \begin{cases} 1.0 & -0.5 < x < 0, \\ 1.0 - (1 - h_m)x^2(2 - x)^2 & 0 < x < 1, \\ h_m & 1 < x < 4. \end{cases}$$

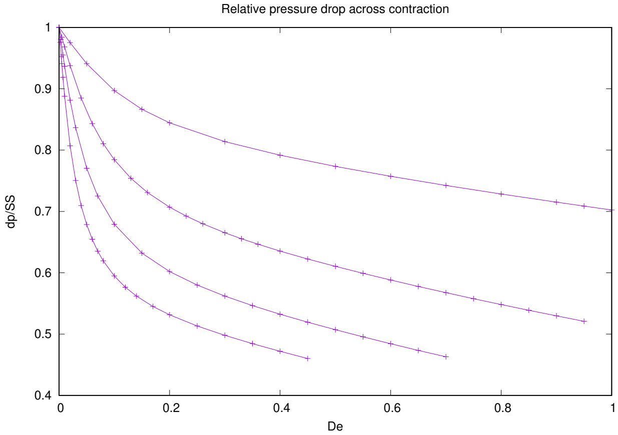


Parameters: $c = 1.0$, $Q = 1$, Contraction to $h_m^{-1} = 2^{1/2}, 2, 2^{3/2}, 4$.

Most calculations have a resolution $\delta x = 0.0208$ and $\delta y = 0.0333$, with random spot checks to confirm around 3-figure accuracy. And $\delta t \approx 10^{-3}$ for stability.

Pressure drop 1, across the contraction, from $x = 0$ to $x = 1$

Divided by the pressure drop for viscous fluid with same steady viscosity, $\mu_0 + G\tau$.



Four curves, from top $h_m^{-1} = 2^{1/2}$, 2, $2^{3/2}$, 4.

Note: typically 30% less pressure drop.

Low De approximation

Examine decrease from steady viscosity result.

Asymptotic result for $De \ll 1$, (Boyko & Stone, JFM 2022)

$$\Delta p \sim -\frac{9De}{2}\left(\frac{1}{h_m^4} - 1\right) + O(De^2) + O(De^3).$$

Low De approximation

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$$\Delta p \sim -\frac{9De}{2}\left(\frac{1}{h_m^4} - 1\right) + O(De^2) + O(De^3).$$

40% from insufficient time for elastic shear stresses to build up to steady values.

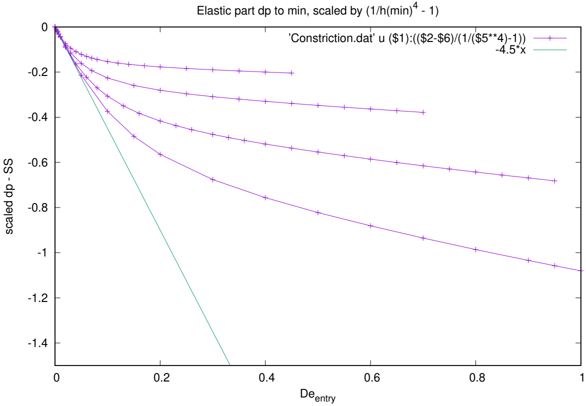
60% from stronger tension in streamlines in narrower channel pulling the flow, so needs less pressure to push.

Nice detail: velocity profile unchanged at $O(De)$ (Giesekus 1963).

Hence plot descrease divided by $(h_m^{-4} - 1)$.

Pressure drop 2, rescaled for $De \ll 1$

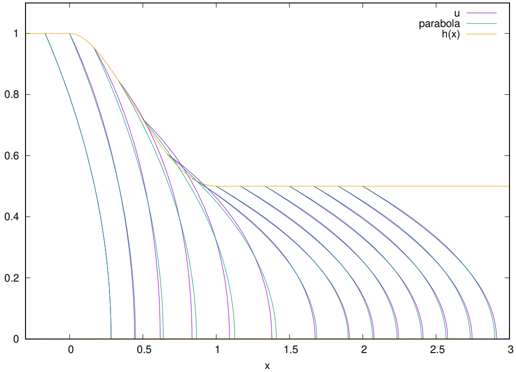
Reduction in pressure drop divided by $(h_m^{-4} - 1)$



Note: low De asymptotics good to $De = 0.1$, significantly off by $De = 0.2$.

Velocity profiles along the flow

*



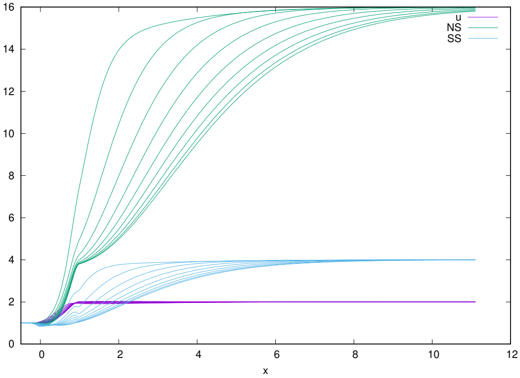
Velocity u
Parabola
$h_m^{-1} = 2$
$De = 0.5$

Very nearly parabolic. Hence will use $\eta = y/h(x) = \mathrm{const.}$ instead of true streamline.

Slightly faster near the wall and slower in the centre, due to gradient in tension in the streamlines.

Streamwise variations, on constant $\eta = 0.1(0.1)0.9$

The velocity, elastic normal stress, and elastic contribution to the shear stress;
all scaled by their entry values ($h_m^{-1} = 2$, $De = 0.5$).



For a contraction of 2 :1,
u increases by 2, SS by 4, NS by 16.

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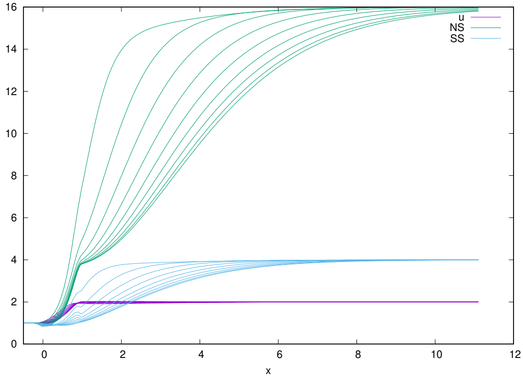


For a contraction of 2 :1, u increases by 2, SS by 4, NS by 16.

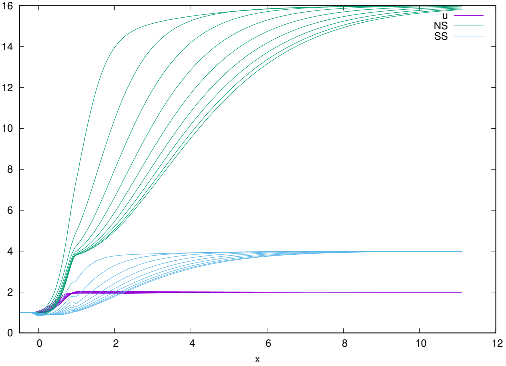
Very long relaxation in centre of channel (lower curves).

Normal stresses achieve 95% steady value at $\Delta x = 8De_{\mathrm{exit}}$.

Numerical advantage of constriction over contraction
(Accelerate numerically with Shanks Transform.)

...streamwise variations, on constant $\eta = 0.1(0.1)0.9$

The velocity, the elastic normal stress, and the elastic contribution to the shear stress; all scaled by their entry values ($h_m^{-1} = 2$, $De = 0.5$).



If need long exit channel for stress relaxation,

then little relaxation in contraction to $x = 1$

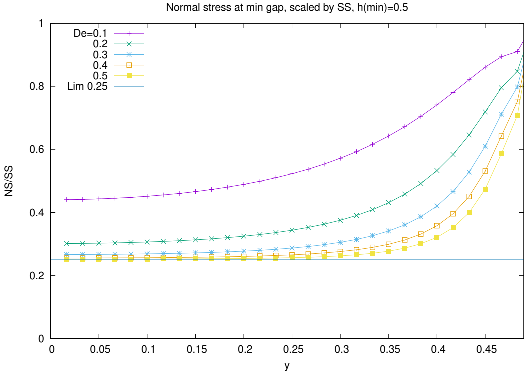
(key to high De asymptotics).

By end of contraction $x = 1$,
shear stress unchanged (not near wall),
normal stresses increased by 4.

Need to examine $x = 1$

Cross-stream variations of normal stress at end of contraction, $x = 1$

Normal stresses at the end of the contraction, divided by value if in steady uniform flow, $\propto h_m^{-4}$.



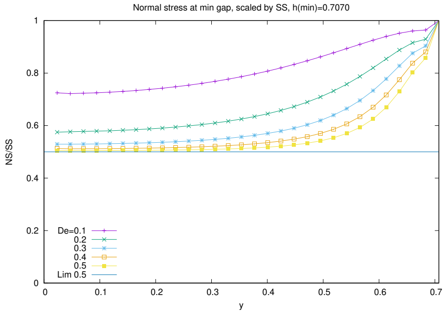
$h_m^{-1} = 2$

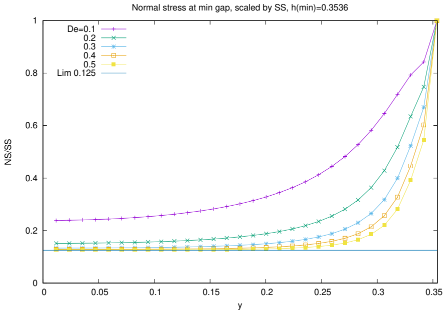
Note tending to limit $0.25 = h_m^2$,

so normal stresses $\propto h_m^{-2}$.

....normal stress across end of contraction

The same for the two cases $h_m^{-1} = 2^{1/2}$ and $2^{3/2}$.





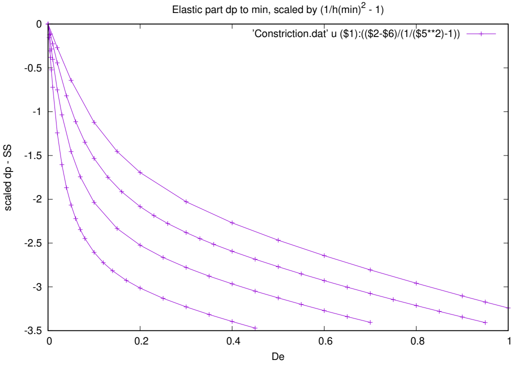
Tending to limits 0.5 and $0.125 = h_m^2$ respectively, so again normal stresses $\propto h_m^{-2}$.

Behaviour well established for $De \geq 0.4$, because little time to relax inside contraction.

Pressure drop 3, towards high *De* theory

At end of contraction, normal stresses increase by factor h_m^{-2}, not uniform flow factor h_m^{-4}.

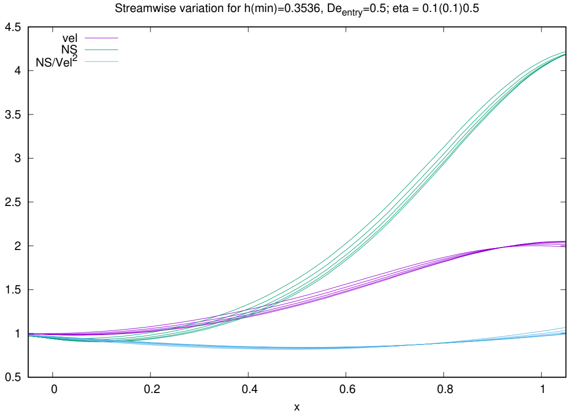
So difference in pull of tension in the streamlines scales with $(h_m^{-2} - 1)$.
Divide the reduction in pressure difference by this.



curves parallel at high *De*

Streamwise variations of normal stresses, on constant $\eta = 0.1(0.1)0.5$

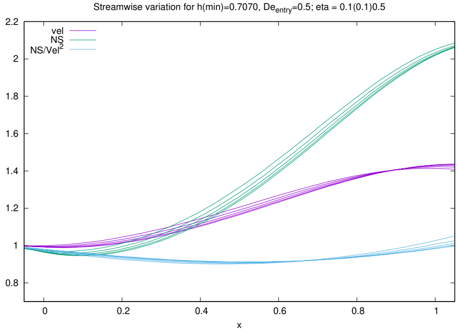
Velocity u, normal stress, and normal stress divided by u^2; all scaled by entry value.
(For $De = 0.5$ and $h_m^{-1} = 2$.)

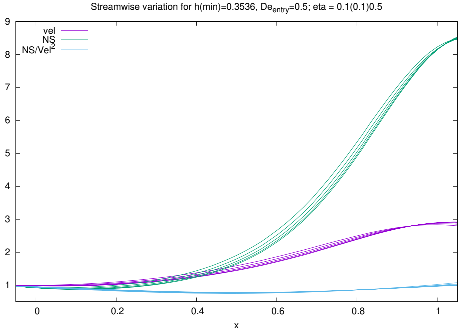


On each streamline, normal stress varies with u^2 along the contraction.

....normal stresses along flow

Same for $h_m^{-1} = 2^{1/2}$ and $2^{3/2}$.





Conclude normal stress varies with u^2.

Explains previous result of the variation across end of contraction limited to h_m^{-2}.

Deforming with the flow 2 – accelerating flow

With no time to relax,

Fibres in direction of flow stretched when flow accelerates, length $\propto u(x)$.

Hence tension in streamlines $\propto u^2(x) \quad \propto h^{-2}(x)$. (ejh 1993)

Fibres perpendicular to flow will be squashed by $u^{-1}(x)$ to conserve volume.

Hence shear stresses (one fibre parallel and one perpendicular) will not change with the flow accelerating.

Basis of high *De* asymptotics.

High De asymptotics

Through contraction, elastic shear stresses keep their value from the entry channel (not increase by h^{-2}), while the normal stresses increase with $h^{-2}(x)$ (not increase by h^{-4}):

$$A_{12} = -\frac{3De}{\cancel{h(x)^2}\,1}\,\eta, \quad A_{11} = \frac{18De^2}{\cancel{h(x)^4}\,h(x)^2}\eta^2.$$

This gives a pressure gradient

$$\frac{dp}{dx} = -\frac{3}{h^3} - \frac{3c}{h} - \frac{18cDe\,h_x}{5h^3}.$$

Integrating

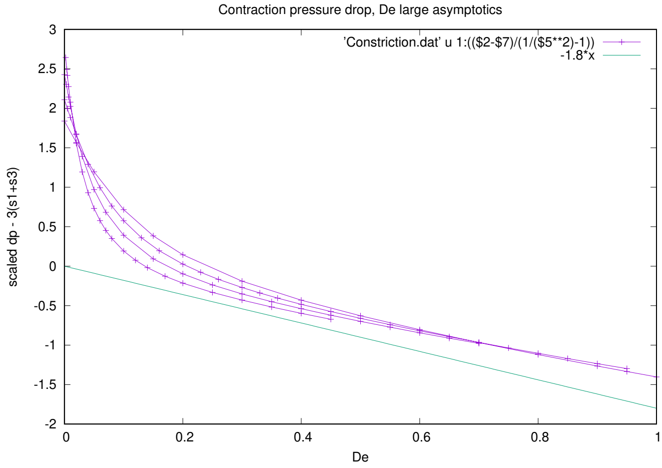
$$\Delta p = -\int_0^1 \left(\frac{3}{h^3} + \frac{3c}{h}\right)\,dx + \frac{9cDe}{5}\left(\frac{1}{h_m^2} - 1\right).$$

The c-term: elastic shear stress without time to achieve steady state, so lower pressure drop.

Second term is stronger tension in streamlines at end of contraction pulling the flow along, so reducing need for pressure to push.

Test of high *De* asymptotics, pressure plot 4

Subtract shear stress terms, then scale with $(h_m^{-2} - 1)$



Test of high De asymptotics, pressure plot 4

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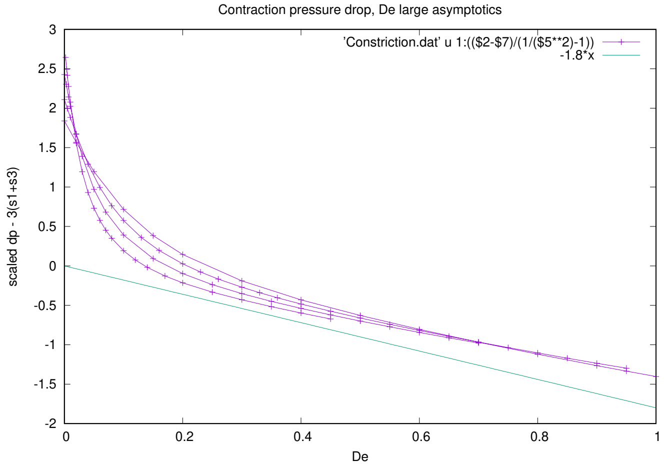
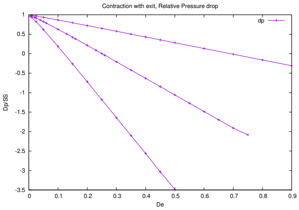
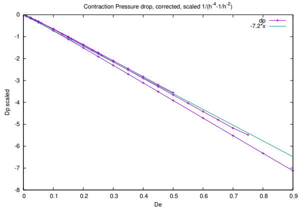
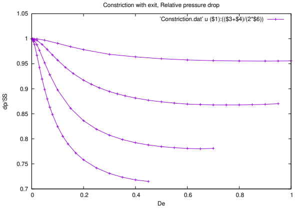
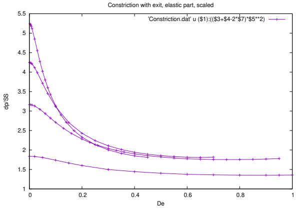


But what about relaxation in exit channel? - work in progress

Pressure drop 5, including contribution from long exit channel

	Relative pressure drop	Elastic part scaled
Contraction		
Constriction		

work in progress

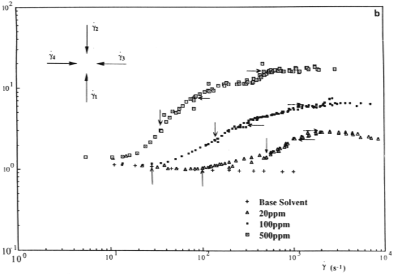
Conclusions: mechanisms

- Visco-elastic Oldroyd-B predicts a reduction in pressure drop compared with a Newtonian viscous fluid with the same steady viscosity
- Some of the reduction comes from elastic stress not having time to reach steady state when moving fast through contraction
- Some of the reduction comes from increased tension in the streamlines pulling the flow through the contraction

Comparison with experiments

Pressure drops for abrupt (i) contraction and (ii) constriction

Saramito & Piau (1994)



Rothstein & McKinley (1999)

76 J.P. Rothstein, G.H. McKinley / J. Non-Newtonian Fluid Mech. 86 (1999) 61–88

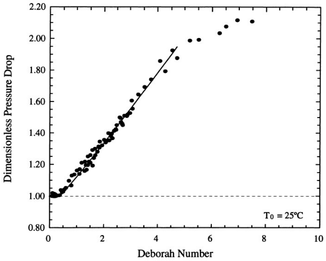


Fig. 7. Dimensionless pressure drop $\mathscr{P}(\mathrm{De}) = \Delta P'_{24}(\mathrm{De}, Q)/\Delta P'_{24}(\mathrm{De}=0, Q)$ of the 0.025 wt.% PS/PS solution across the 4 : 1 : 4 axisymmetric contraction as a function of Deborah number $\mathrm{De} = \lambda_0\dot{\gamma}$. The figure includes: ●, experimental data; ———, linear regression of the form $\mathscr{P} = 1 + 0.2(\mathrm{De} - \mathrm{De}_{\mathrm{cr1}})^{1.05}$.

Pressure drop increases in experiments.

Comparison with experiments

Wrong sign!

Experiments have abrupt contraction, with upstream vortices

Unavoidable: reduction due to time delay to achieve steady state.

Unavoidable: reduction due to higher tension in streamlines pulling flow into contracted channel,
but would less if normal stresses stopped increasing quadratically with shear-rate.

Need extra dissipation in the accelerating flow: an extensional viscosity (say from FENE), or more subtly internal modes dissipating (ejh 1994).

Thanks

To the organisers, To the participants,

Thanks

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