A perspective of Batchelor's Micro-hydrodynamics

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October 19, 2016

A change of research direction, a big change

Turbulence – first research career 1941–1961.

Stopped completely.

Micro-hydrodynamics – second research career 1967–1986,

producing 8 of his top 10 most cited papers.

GKB's Micro-hydrodynamics in perspective

- Before
  - Stokes drag $6\pi\mu a V$
  - Einstein viscosity $\mu(1+\frac{5}{2}c)$
  - Einstein diffusivity $D = kT/6\pi\mu a$
  - GITayor – drops and viscosity of emulsion
  - Brenner, Cox, Mason, Giesekus, Saffman, Bretherton
- Batchelor's research
  - Hydrodynamic corrections to Stokes & Einstein
  - To start: the bulk stress
- What followed

A start

Stress system in a suspension of force-free particles JFM 1970

Suspension of small particles in a viscous fluid

- low Reynolds number flow about particles
- each particle sees a linear flow (local rheology)

Everywhere

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij} + \sigma^{+}_{ij}$$

with viscous solvent stress and extra non-zero only inside particles

Ensemble average (from turbulence research)

$$\langle \sigma_{ij} \rangle = -\langle p \rangle \delta_{ij} + 2\mu \langle e_{ij} \rangle + \langle \sigma^{+}_{ij} \rangle$$

Switch to volume average. Use divergence theorem for force-free particles

$$\langle \sigma_{ij}^{+} \rangle = n \int_A \sigma_{ik} n_k x_j - \mu(u_i n_j + u_j n_i)\, dA$$

with n number of particles per unit volume and A surface of typical particle.

L&L 1959

. . . a start

Stress system in a suspension of force-free particles JFM 1970

What followed?

- 880+ citations.
- Evaluation of bulk stress in many rheologies,

  once solved micro-structure evolution
- Ensemble average needed for calculating non-local rheology
- Homogenisation (1980s) – a step back,

  except compute in periodic boxes

Second result

Stress generated in a non-dilute suspension of elongated particles by pure straining motion JFM 1971

A suspension of fibres.

Rods of length ℓ, radius b, number per unit volume n.
Hence average lateral spacing $h = (2n\ell)^{-1/2}$.

Regimes

- (Very) dilute: $\ell \ll h$
- Semi-dilute: $b \ll h \ll \ell$
- (Nematic phase transition to aligned rods: $h = \sqrt{b\ell}$)
- Concentrated: $h \sim b$

...second result

Stress generated in a non-dilute suspension of elongated particles by pure straining motion JFM 1971

Use bulk-stress paper for formula for stress
Use slender-body paper, with outer boundary condition at typical separation h in place of at length ℓ

Result: extensional viscosity

$$\mu_{\text{ext}} = \mu \frac{4\pi n \ell^3}{9 \log h/b}$$

One data point

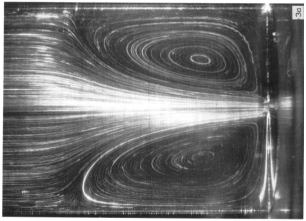
Much larger than the viscosity of the solvent μ, which is shear viscosity of suspension

. . . second result

Stress generated in a non-dilute suspension of elongated particles by pure straining motion JFM 1971

What followed?

- Large effect from small concentration $\rightarrow$ TDR
- Correct outer with a Brinkman approach by Shaqfeh (1990)
- Anisotropic viscosity ($\mu_{\text{ext}} \gg \mu_{\text{shear}}$) produces anisotropic flow



- Anistropic flow $\rightarrow$ TDR

Renormalization of hydrodynamic interactions

Sedimentation in a dilute suspension of spheres JFM 1972

Settling velocity of test sphere due to 2nd at distance r

$$V(r) = V_0 + \Delta V(r)$$

with Stokes velocity for isolated sphere $V_0 = 2\Delta\rho g a^2/9\mu$

Far field form from reflections

$$\begin{aligned} \Delta V(r)/V_0 \quad = \quad & \tfrac{a}{r} + \tfrac{a^3}{r^3} && \text{1st reflection} \\ & + \tfrac{a^4}{r^4} + \tfrac{a^6}{r^6} + \dots && \text{2nd reflection} \\ & + \tfrac{a^7}{r^7} + \tfrac{a^9}{r^9} + \dots && \text{3rd reflection} \\ & + \dots \end{aligned}$$

ignoring directional dependency

...renormalization of hydrodynamic interactions

Sedimentation in a dilute suspension of spheres JFM 1972

Naive pairwise addition of disturbances within large domain $r \leq R$, with n spheres per unit volume

$$\langle \Delta V \rangle = \int_{r=2a}^{R} V_0 \left(\frac{a}{r} + \dots \right) n \, dV = O\left(V_0 c \frac{R^2}{a^2} \right)$$

$c = \frac{4\pi}{3} n a^3$ the volume fraction.

The divergence problem:

- Does mean settling velocity depends on size of domain R?
- Or is it an intrinsic property independent of domain?
  i.e. is pairwise addition naive?

... renormalization of hydrodynamic interactions

Sedimentation in a dilute suspension of spheres JFM 1972

Batchelor's renormalization:

$$\Delta V = \left(1 + \tfrac{a^2}{6}\nabla^2\right) u(x)\big|_{\text{test sphere}} + \text{higher reflections}$$

Pairwise sum of $O(V_0 a^4/r^4)$ higher reflections is convergent

Now

$$\langle u \rangle_{\text{everywhere}} = 0, \qquad \text{so}$$

$$\langle u \rangle_{\text{test sphere}} = -\tfrac{11}{2} V_0 c \qquad \textit{back flow}$$

$$\langle \tfrac{a^2}{6}\nabla^2 u \rangle_{\text{test sphere}} = \tfrac{1}{2} V_0 c$$

$$\langle \text{higher reflections} \rangle = -1.55 V_0 c$$

Hence

$$\langle V \rangle = V_0(1 - 6.55c)$$

...renormalization of hydrodynamic interactions

Sedimentation in a dilute suspension of spheres JFM 1972

What followed?

- 910+ citations
- Erroneous applications subtracting wrong infinity
- Alternative *averaged equation* approach
  recognising divergences as change of ρ and μ
  from solvent to suspension values
- Much more from Batchelor, e.g.

Brownian diffusion of particle with hydrodynamic interactions JFM 1976

$$D = \frac{1 - 6.55c}{6\pi\mu a}\left[\frac{c}{1-c}\frac{\partial\mu}{\partial c} = kT(1 + 8c + 30c^2)\right]$$

$O(c^2)$ correction to Einstein viscosity

The determination of the bulk stress in a suspension of spherical particles to order c^2
JFM 1972

Another renormalization of hydrodynamic interactions, with J.T Green (PhD 1970, without answer)

For pure straining, by trajectory calculation of nonuniform probability distribution of separation of pairs

$$\mu^* = \mu \left(1 + \tfrac{5}{2}c + 7.6c^2\right)$$

For simple shear, problem of closed trajectories ($k = 5.2$?)

$$\mu^* = \mu \left(1 + \tfrac{5}{2}c + kc^2\right)$$

Case of strong Brownian motion (JFM 1977)

$$\mu^* = \mu \left(1 + \tfrac{5}{2}c + 6.2c^2\right)$$

Note strain-hardening and shear-thinning rheology

A refinement - polydispersity

Sedimentation in a dilute polydisperse system of interacting spheres JFM 1982, Parts I, II, Corrigendum

$$\langle V_i \rangle = V_{i\,0}\,(1 + S_{ij} c_j)$$

If equal density and nearly equal sizes, then increase of near pairs and

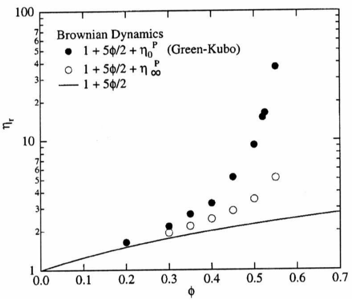
$$\langle V \rangle = V_0\,(1 - 5.6c)$$

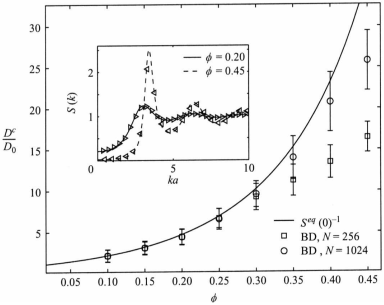
as in experiments and dilute limit of Richardson-Zaki.

Also
Diffusion in a dilute polydisperse system of interacting spheres
JFM 1983

What followed beyond Batchelor's c^2

Brady's Stokesian Dynamics (1988) numerical simulations at moderate concentrations of c1000 hard spheres





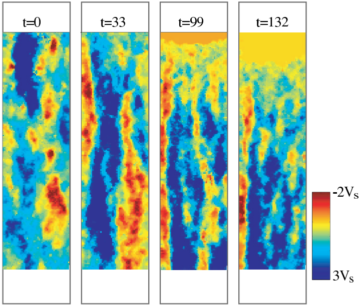
Boundary integral methods for emulsions (1996) c12 drops

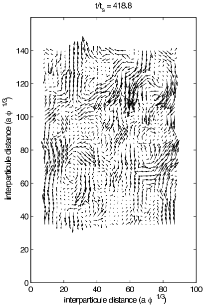
Ladd's Lattice Boltzmann simulations (1996) c32000 particles

What followed in sedimentation

- Inclined settling (Boycott effect) Acrivos 1979
- Structural instability for fibres Koch & Shaqfeh 1989
- Fluctuations depend on the size of box Guazzelli 2001

$$\langle V'^2 \rangle = O\left(V_0^2 c \frac{R}{a}\right)$$





Suspensions: what happened in parallel

- Orientation of non-spheres, with Brownian motion

  Leal & Hinch 1972
- Deformation and breakup of drops, emulsions   Acrivos 1970
- Electrical Double Layers and VdW forces   Russel 1978
- Rough spheres   Leighton 1989

What followed

- Rheology, with experiments
- Fluid dynamics of non-Newtonian fluids

  Normal stresses, stress relaxation, stress saturation, elastic boundary layers, anisotropic flow
- Electro- and Magneto- rheological fluids
- Microfluidic devices – exploiting/ignoring techniques

  drop production, mixing, slip at wall
- Micro-rheology

A change of research direction, a big change

Turbulence – first research career 1941–1961.

- Understand, explain and exploit Kolmogorov
- Townsend's experiments
- Troubled cannot solve Navier-Stokes
- Marseille 1961 IUTAM/IUGG Congress

Time off: JFM, textbook

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producing 8 of his top 10 most cited papers.