A load of balls in Newton's cradle

or Fragmentation of a line of balls by an impact

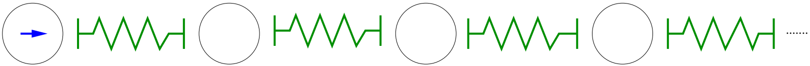
John Hinch

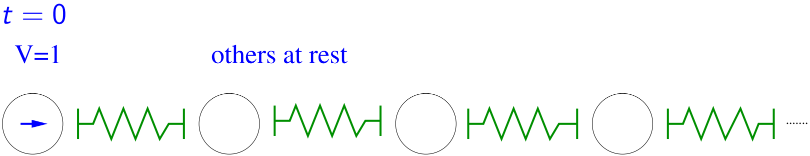
DAMTP, Cambridge

March 26, 2010

$t = 0$

V=1 others at rest



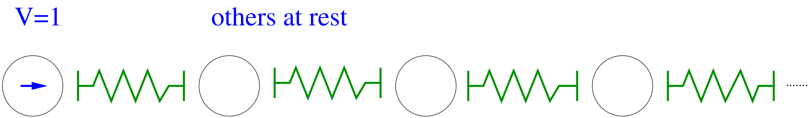


Contacts: linear springs in compression, zero force in tension

$$\ddot{x}_n = (x_{n+1} - x_n)_- - (x_n - x_{n-1})_-$$

No non-dimensional groups

$t = 0$



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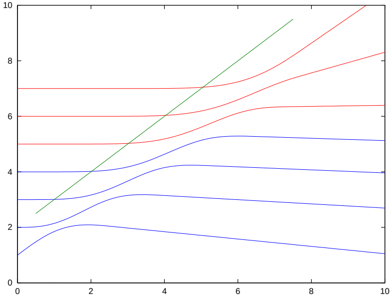
Granular media?

Position of particles

$n + x_n$

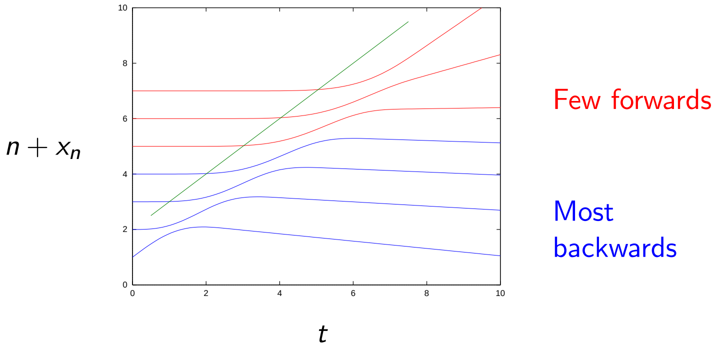


t

Few forwards

Most backwards

Position of particles



If in contact: $\ddot{x}_n = x_{n+1} - 2x_n + x_{n-1}$

Continuum approximation: $x_{tt} = x_{nn}$, with wave speed 1.

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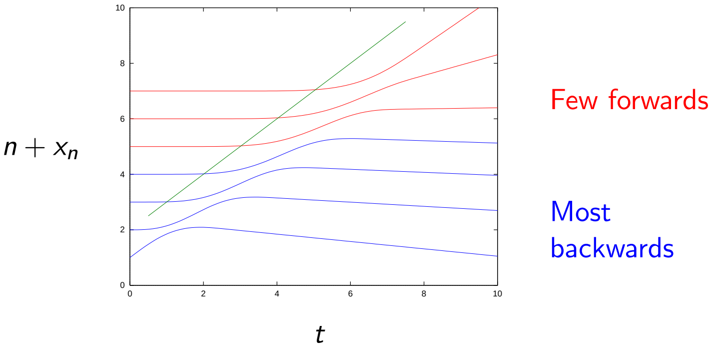

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Non-continuum effect: $\dot{x}_n(t = \infty) < 0$

Questions

- How many fly off at the far end?

  At what velocities?

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Simple mechanics

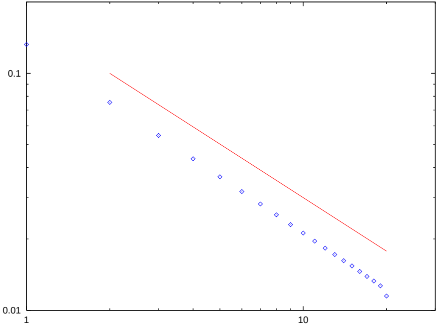
Simple questions

Answers more complicated

Rebound velocities

Chain of 25 particle, 20 rebound

$-\dot{x}_n(\infty)$



n

$n^{-3/4}$?

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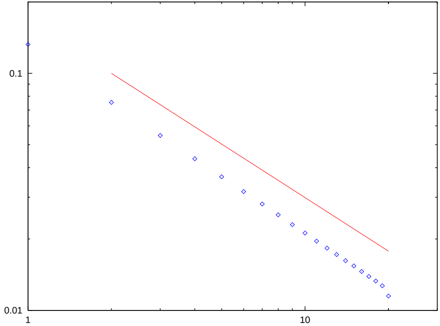

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Rebound energy $\sum n^{-3/2}$ is finite
Rebound momentum $\sum n^{-3/4}$ is infinite

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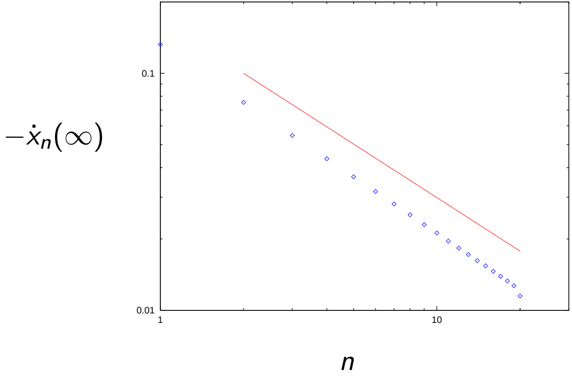
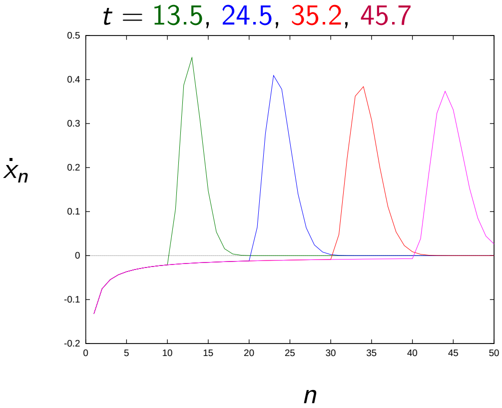


$n^{-3/4}$? If so

Rebound energy $\sum n^{-3/2}$ is finite
Rebound momentum $\sum n^{-3/4}$ is infinite

But why $n^{-3/4}$? Not diffusion.

Key: propagation of an impulse wave



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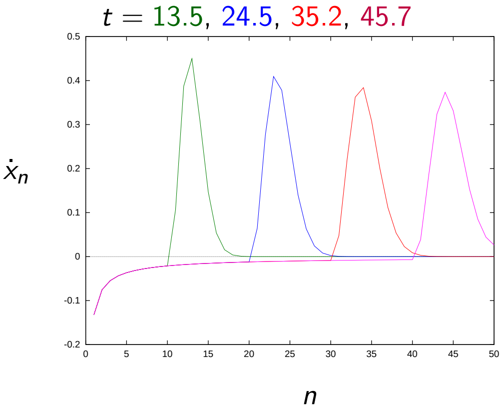

Peak velocity decreases slowly. How?

Width of pulse increases slowly. How?

Spreading wave which conserves energy

Slowly varying amplitude and wavelength of propagation wave of constant form $f(.)$

$$x_n = a(t)\, f\left(\frac{n-t}{\lambda(t)}\right)$$

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Replot for different t, $\dot{x}_n a$ against $(n-n_1)/a^2$, where n_1 is last in contact, and $a = x_{n_1}$.

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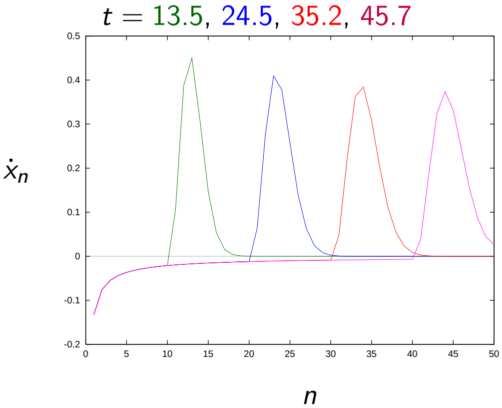

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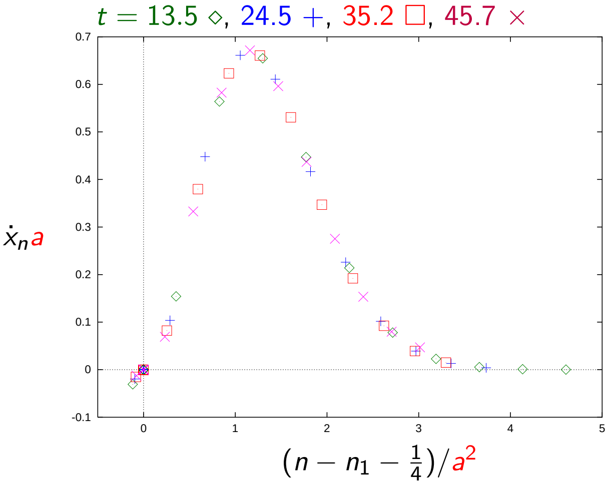
Self similar impulse wave



Here $a = x_{n_1}$ at last in contact.

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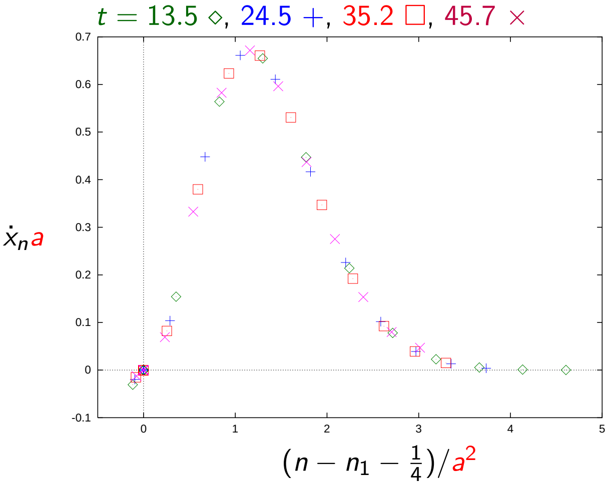

Here $a = x_{n_1}$ at last in contact. But how to predict a(t)?

Scaling for spreading wave : $\lambda(t)$?

If touching

$$\ddot{x}_n = x_{n+1} - 2x_n + x_{n-1}$$

Second approximation for continuum limit

$$x_{tt} = x_{nn} + \frac{1}{12}x_{nnnn}$$

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To balance last two terms, use similarity variable $N/t^{1/3}$.

Hence $\lambda \propto t^{1/3}$

Results for spreading wave

$$\begin{array}{llll} \text{wavelength} & & \lambda & \propto t^{1/3} \\ \text{displacements} & x_n = & a & \propto t^{1/6} \\ \text{velocities} & \dot{x}_n = & a/\lambda & \propto t^{-1/6} \end{array}$$

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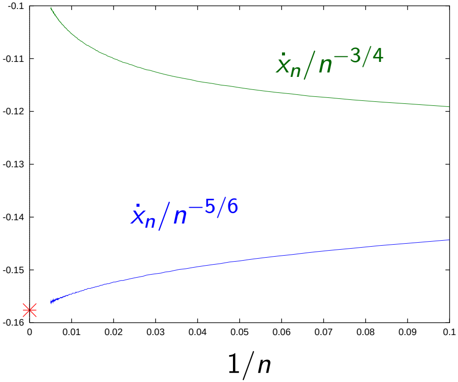
Δt time between particles rebounding $= 1$ (wave speed $= 1$)

Hence rebound velocities

$$\dot{x}_n(\infty) \propto -t^{-5/6} = -n^{-5/6}$$

Rebound velocities



$$\dot{x}_n(\infty) = -0.158 n^{-5/6}$$

Similarity solution

Try $\quad x(n,t) = t^{1/6} f\left(\xi = \frac{n-t}{t^{1/3}}\right) \quad$ in $\quad x_{tt} = x_{nn} + \frac{1}{12} x_{nnnn}$

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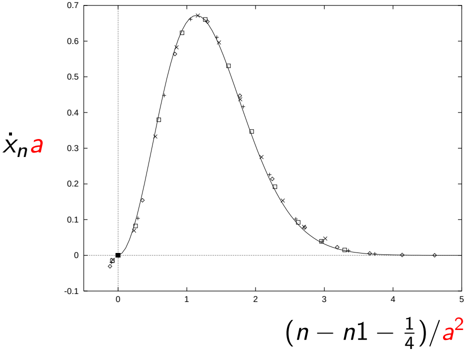

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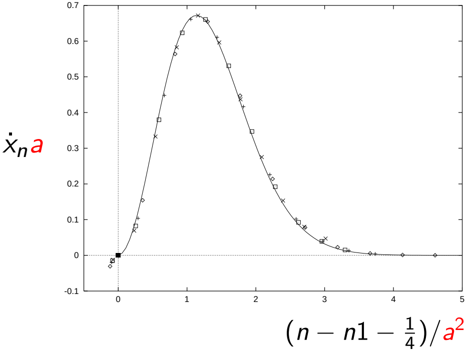

Solution $\quad f = \int_\xi^\infty \mathrm{Ai}^2(-2^{1/3} y)\, dy$

The $\frac{1}{4}$ shift

Near to back of the wave $\xi = \xi_0$

$$f \sim f(\xi_0)\left(1 - \tfrac{2}{3}(\xi - \xi_0)^3 + \dots\right)$$

Correction for ejected velocities at $t^{-5/6}$

$$x_n(t) \sim t^{1/6} f(\xi) \; + \; t^{-1/2}\beta(\xi - \xi_0)$$

Ball n detaches at t_n where $\xi = \xi_0 + \delta$ if $x_n(t_n) = x_{n+1}(t_n)$, i.e.

$$t^{-5/6}\left[f(\xi_0)\left(2\delta^2 + 2\delta - \tfrac{2}{3}\right) - \beta\right] = 0$$

and detaches with known velocity

$$-\tfrac{1}{6} f(\xi_0) t^{-5/6} = \dot{x}_n = t^{-5/6}\left[f(\xi_0) 2\delta^2 - \beta\right]$$

Hence $\delta = \frac{1}{4}$.

Finite chain of N: number fly off and their velocities

When wave reaches end at $t = N$.
width of wave $1.5N^{1/3}$ and velocities $1.4N^{-1/6}$ and less.

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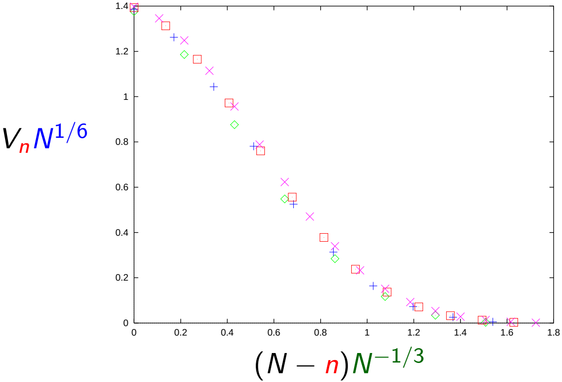
$N = 100 \diamond$, $200 +$, $400 \square$, $800 \times$



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Simple questions

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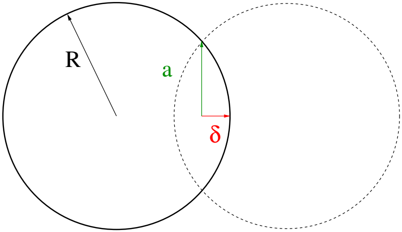
For linear force law. Next nonlinear Hertz contacts

Hertz contacts



Radius R
Contact radius a
Overlap δ

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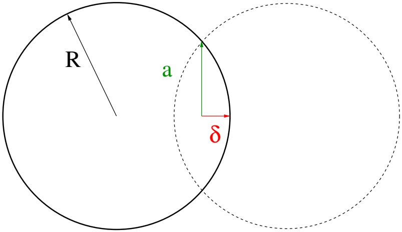

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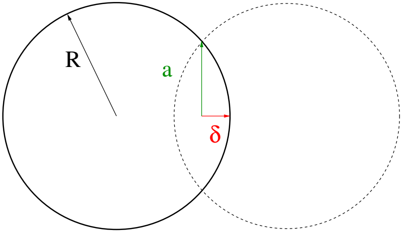

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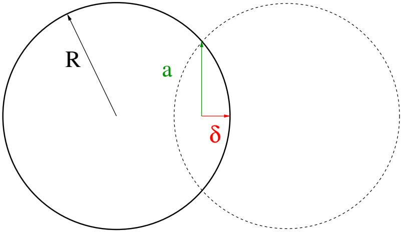

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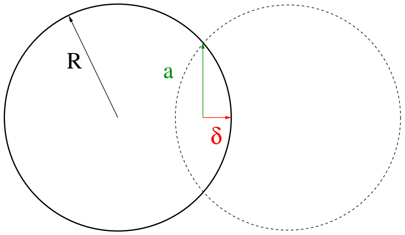

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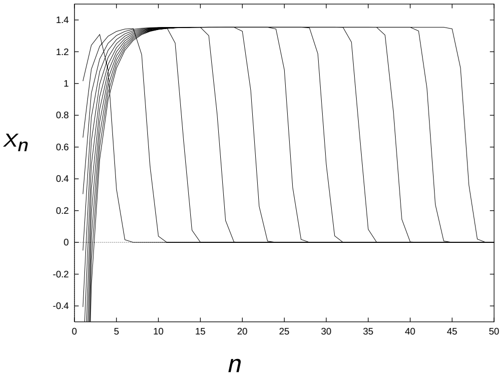
$$\text{Force} = \pi a^2 E\frac{\delta}{a} = \frac{\sqrt{2}E}{3(1-\nu^2)}R^{1/2}\delta^{3/2}$$

Impulse wave propagating down a Hertzian chain



No spreading: nonlinearity balances ‘diffusion’

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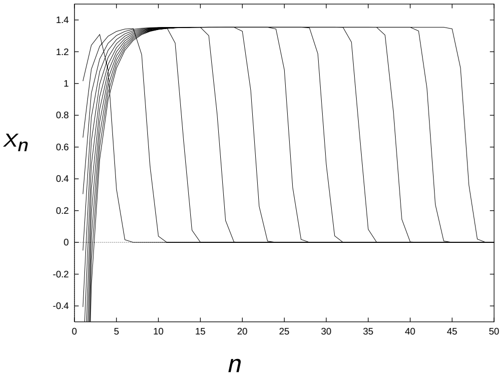

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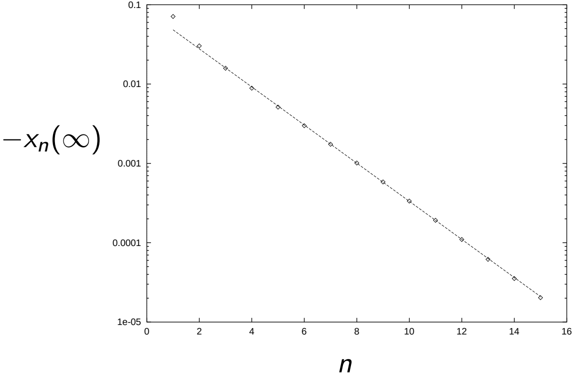
Two fly off end: $V_N = 0.984$, $V_{N-1} = 0.149$,
$(V_{N-2} = 0.004,\ V_{N-3} = 10^{-5},\ V_{n-4} = 3\times10^{-8})$

Rebound velocities for Hertzian chain



$$x_n(t = \infty) \; = \; -0.084 e^{-0.55 n}$$

Nesterenko's (1984) solitary wave (long wave approx)

$$\ddot{x} = D\left[(Dx)^{3/2}\right] \quad \text{where} \quad Du = u_{n+\frac{1}{2}} - u_{n-\frac{1}{2}} \sim \frac{\partial u}{\partial n} + \frac{1}{24}\frac{\partial^3 u}{\partial n^3}$$

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Predict	$V = 0.896$	Max $\dot{x}_n = 0.902$	$[x_n] = 1.875$
Numerical	$V = 0.841$	Max $\dot{x}_n = 0.681$	$[x_n] = 1.354$

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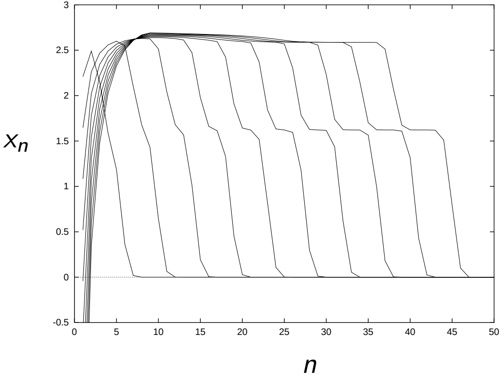
Wave not so long with just 4 balls

Finished?

Impact by two

Two solitary waves



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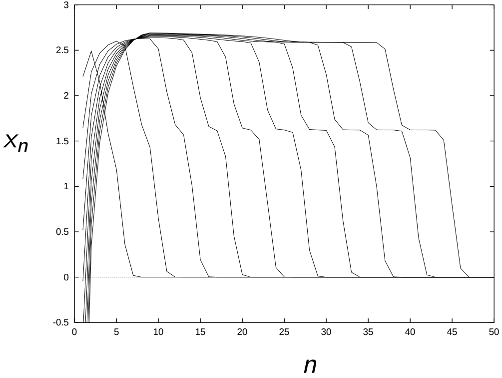


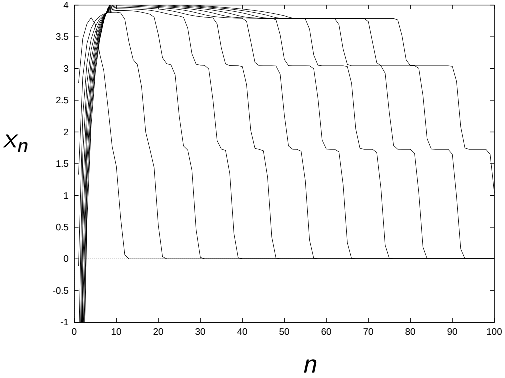
Balls fly of end:

from first wave at 1.234, 0.186,

from second wave at 0.647, 0.090

Impact by three

Three solitary waves



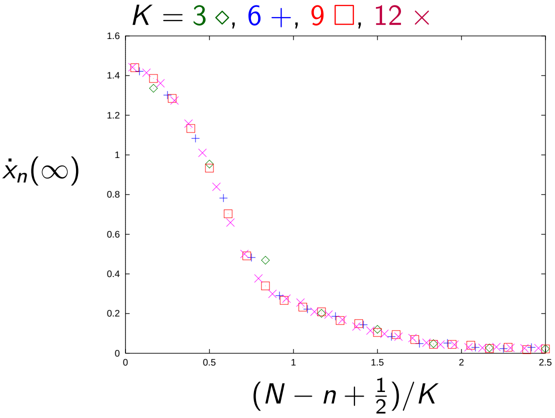
From first wave at 1.336, 0.203,

second at 0.951, 0.141,

third at 0.469, 0.054

Impact by K

$2K$ fly off at far end

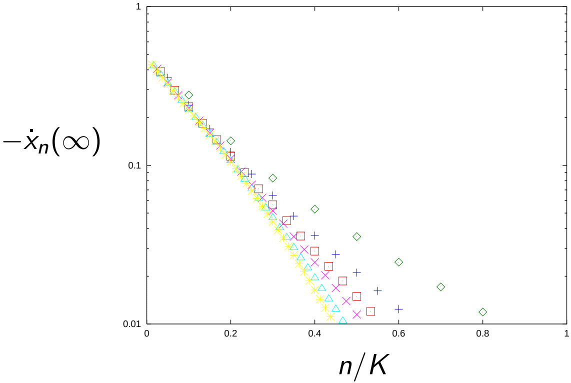


Open question

Impact by K

$\approx \frac{1}{2}K$ rebound at velocities > 0.01.

$K = 10 \diamond,\ 20 +,\ 30 \square,\ 40 \times,\ 60 \triangle,\ 80 *$



Open question

Newton's cradle

Final velocities of position n after impact by K

n	$K = 1$	$K = 2$	$K = 3$	$K = 4$
1	-0.0711	-0.1126	-0.1397	0.0112
2	-0.0303	-0.0420	0.1996	0.8729
3	-0.0145	0.2145	0.7855	1.0145
4	0.1270	0.8004	1.0420	1.0303
5	0.9888	1.1397	1.1126	1.0711

Extra forward

Granular media

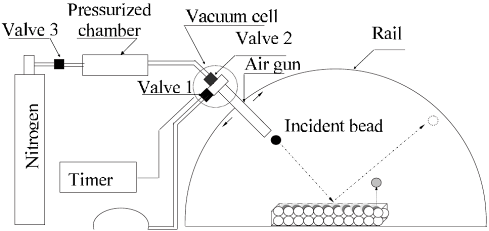
Granular media

Barchan dunes

Aeolian sand transport

Splash experiment in Rennes

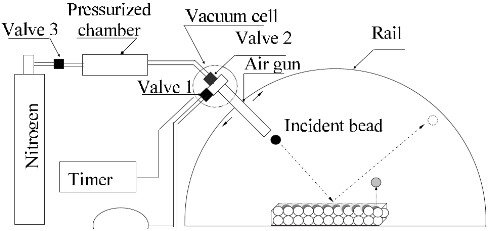
Oger & Valance



Apparatus

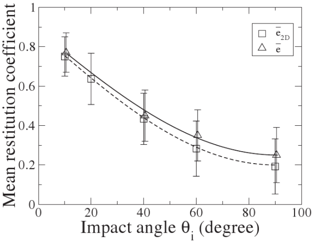
Splash experiment in Rennes

Oger & Valance



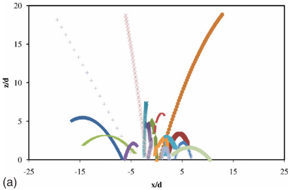
Apparatus

Coefficient of restitution



Splash experiment in Rennes

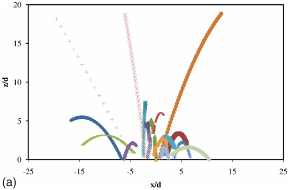
Oger & Valance



Splash of particles

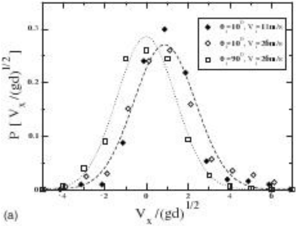
Splash experiment in Rennes

Oger & Valance



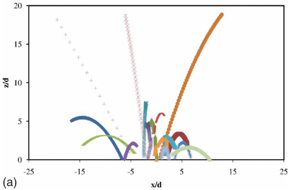
Splash of particles

$V' = 2\sqrt{gd}$, all V_i, θ_i

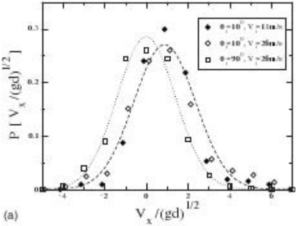


Splash experiment in Rennes

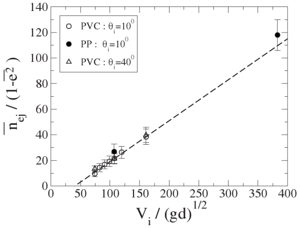
Oger & Valance



Splash of particles



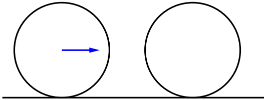
$V' = 2\sqrt{gd}$, all V_i, θ_i



$\#$ ejected $\propto V_i$

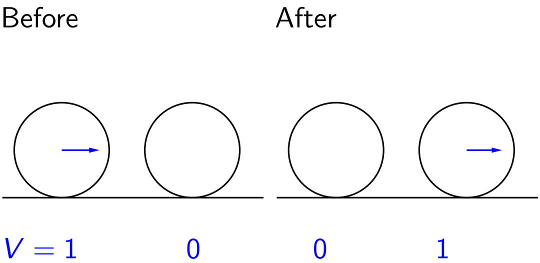
Spin in billiards

Before



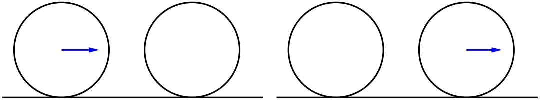
$V = 1$ $\qquad$ 0

Spin in billiards



Spin in billiards

Before After

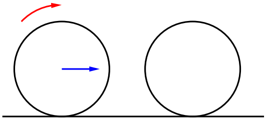


$V = 1$ 0 0 1

But spinning when rolling

Spin in billiards

Before

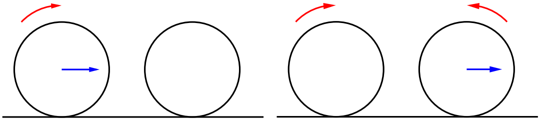


$V = 1$ $\quad$ 0

$\omega = -1$ $\quad$ 0

Spin in billiards

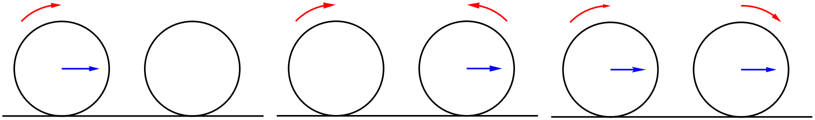
Before   After if $\mu > 0.2$



$V = 1$	0	0	1
$\omega = -1$	0	-0.5	0.5

Spin in billiards

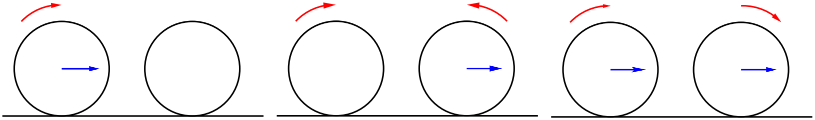
Before | After if $\mu > 0.2$ | Later, rolling again



$V = 1$	0	0	1	$\frac{1}{7}$	$\frac{4}{7}$
$\omega = -1$	0	-0.5	0.5	$-\frac{1}{7}$	$-\frac{4}{7}$

Spin in billiards

Before		After if $\mu > 0.2$		Later, rolling again	

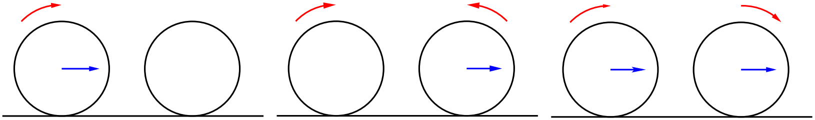


$V = 1$	0	0	1	$\frac{1}{7}$	$\frac{4}{7}$
$\omega = -1$	0	-0.5	0.5	$-\frac{1}{7}$	$-\frac{4}{7}$

$\frac{\text{Energy}}{\frac{1}{2}mV^2} = \frac{7}{5}$ $\qquad$ $\frac{6}{5}$ (86%) $\qquad$ $\frac{17}{35}$ (35%)

Spin in billiards

Before		After if $\mu > 0.2$		Later, rolling again	



$V = 1$	0	0	1	$\frac{1}{7}$	$\frac{4}{7}$
$\omega = -1$	0	-0.5	0.5	$-\frac{1}{7}$	$-\frac{4}{7}$

$$\frac{\text{Energy}}{\frac{1}{2}mV^2} = \frac{7}{5} \qquad \frac{6}{5}\ (86\%) \qquad \frac{17}{35}\ (35\%)$$

Energy lost through spinning rather than restitution

Newton's cradle

Final velocities of position n after impact by K

n	$K = 1$	$K = 2$	$K = 3$	$K = 4$
1	-0.0711	-0.1126	-0.1397	0.0112
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Extra forward