Inclined to exchange: shocking gravity currents

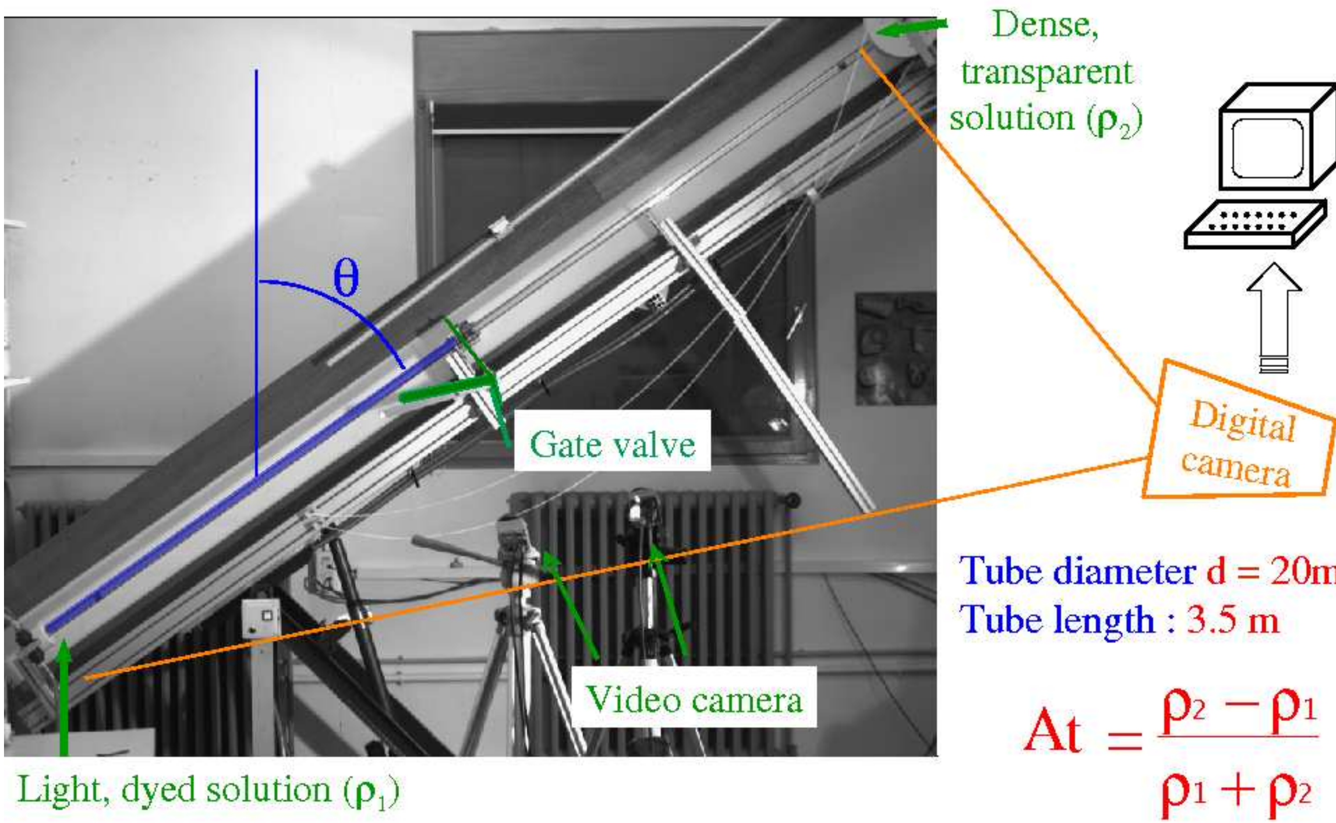
John Hinch (DAMTP)

and at FAST/Paris:

Jean-Pierre Hulin, Dominique Salin & Bernard Perrin

Thomas Séon & Jemil Znaien

Experimental setup

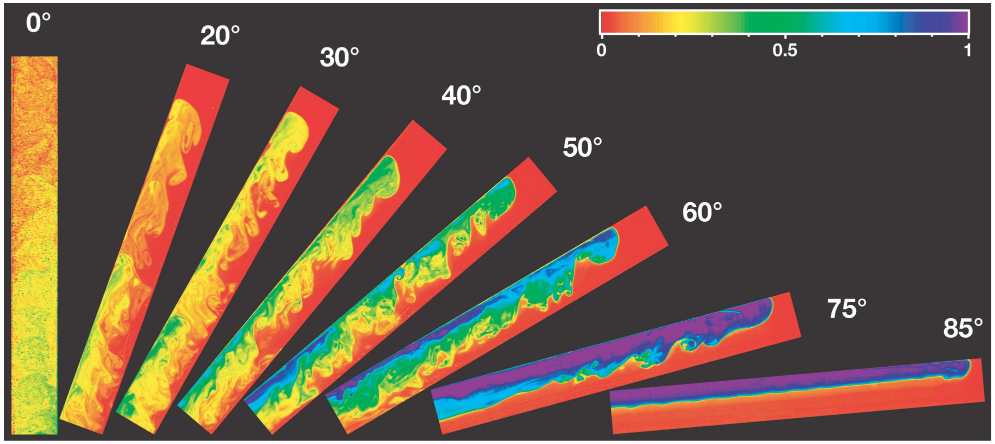


Tube diameter d = 20mm
Tube length : 3.5 m

$$At = \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}$$

Water-$CaCl_2$ solutions : $CaCl_2$: 1 to 100 g/l $\Rightarrow$ At = 4 10^{-4} – 3.5 10^{-2}

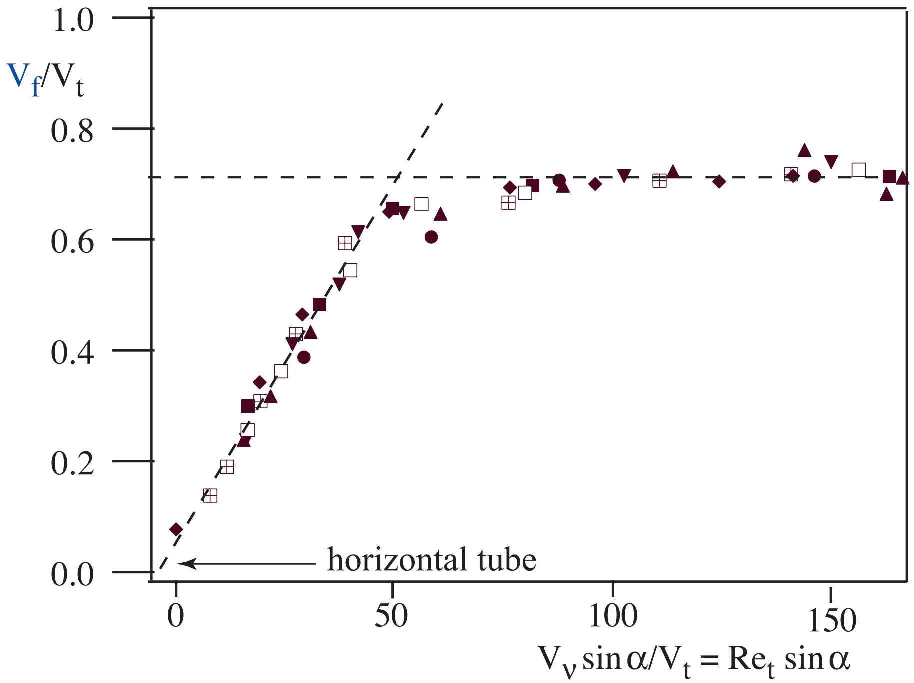
Regimes



- Vertical : turbulent
- Inclined : nosed controlled gravity current
- Horizontal : viscous counter-current

Speed of front V_f

$$V_t = \sqrt{Atgd} \quad \text{(inertial nose)} \qquad V_\nu = \frac{Atgd^2}{\nu} \quad \text{(viscous)}$$



Tilt angles $0 < \alpha < 30^o$; $At = 3.5 \times 10^{-2}$ (●), 10^{-2} (■), 4×10^{-3} (♦), 10^{-3} (▼), 4×10^{-4} (▲) for a viscosity $\mu = 10^{-3}$ Pa.s; and $\mu = 10^{-3}$ Pa.s (■), 4×10^{-3} Pa.s (⊞) for a density contrast $At = 10^{-2}$

$$V_f \approx 0.014 V_\nu \sin\alpha$$

A little theory

Counter current in a circle at 50% fill

$$\nabla^2 u = f(\theta) = \begin{cases} -\frac{1}{2} & \text{in } 0 \leq \theta < \pi, \\ +\frac{1}{2} & \text{in } \pi \leq \theta < 2\pi \end{cases}$$

and $u = 0$ on $r = 1$.

Fourier series

$$f(\theta) = -\sum_{n \text{ odd}} \frac{2}{\pi n} \sin n\theta,$$

so

$$u = \sum_{n \text{ odd}} \frac{2}{\pi n(n^2 - 4)} (r^2 - r^n) \sin n\theta.$$

a little theory continued

Hence flux

$$
\begin{aligned}
Q &= \sum_{n \text{ odd}} \frac{1}{\pi n^2 (n+2)^2} \\
&= \frac{\pi}{16} - \frac{1}{2\pi} \\
&= 0.037195
\end{aligned}
$$

Front velocity for area $A = \frac{\pi}{2}$

$$V_f = \frac{Q}{A} = 0.012,$$

experiment 0.014 – eventually will do better.

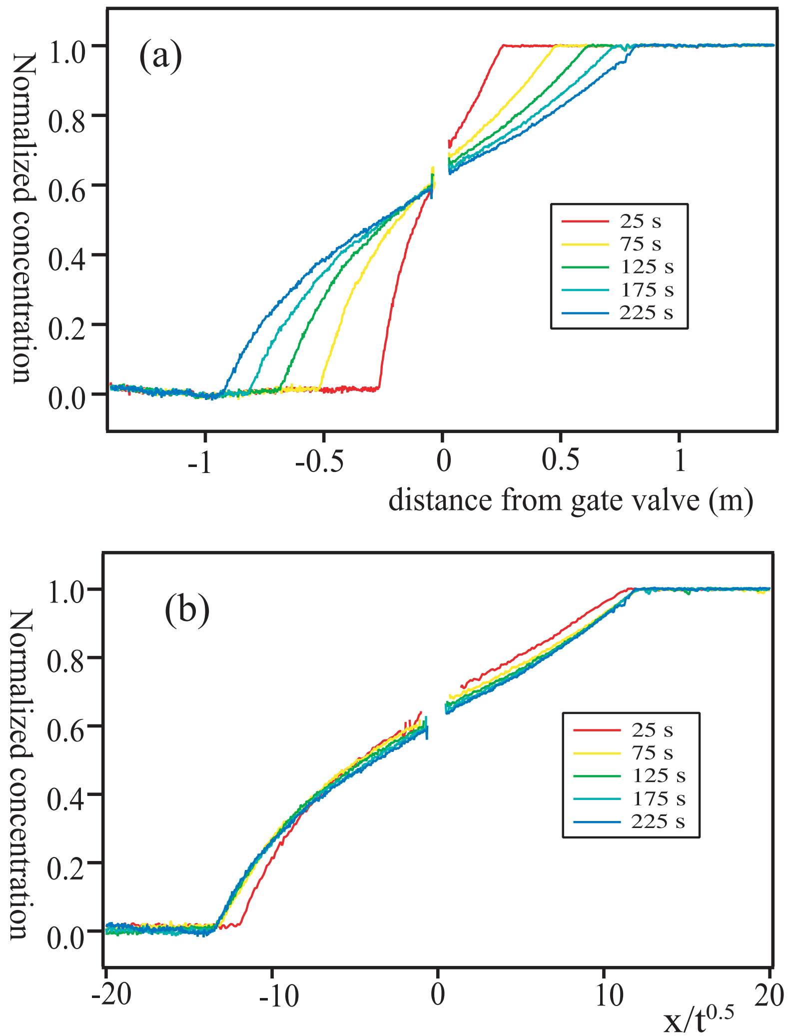
Horizontal tube

$$\begin{aligned} \dot{X}_f &= V_f \\ &\propto \frac{R^2}{\mu}\frac{dp}{dx} \\ &\propto \frac{R^2}{\mu}\frac{\Delta\rho g R}{X_f} \end{aligned}$$

Hence

$$X_f = \sqrt{Dt}$$

$D = 0.0108 V_\nu d$ in experiment



2D spreading in a horizontal *channel*

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{dp_\pm}{dx} \quad \text{in } 0 \le y \le h \text{ and } h \le y \le a,$$

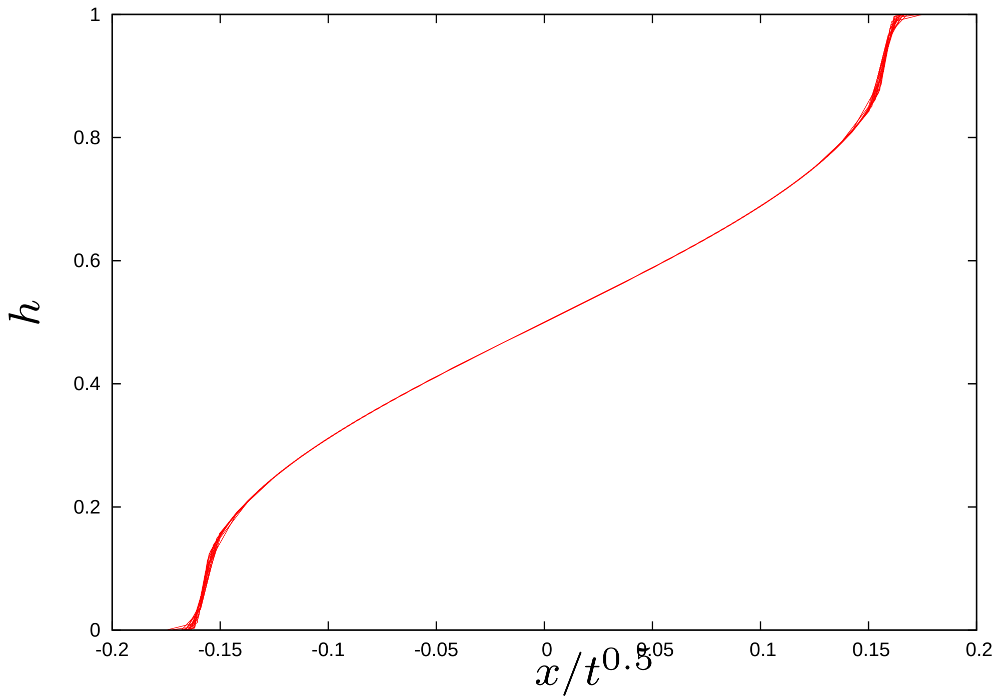
with $u = 0$ on $y = 0$ and a.

No net flux $\quad \int u\, dy = 0 \quad$ and $\quad \left[\frac{dp}{dx}\right]_-^+ = \Delta\rho g \frac{\partial h}{\partial x}.$

Hence

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\Delta\rho g h^3 (a-h)^3}{3\mu a^3}\frac{\partial h}{\partial x}\right).$$

Spreading in a horizontal *channel*, continued



Hence

$$X_f = \sqrt{Dt}$$

with $D = 0.017 V_\nu a$, cf experiment 0.0108.

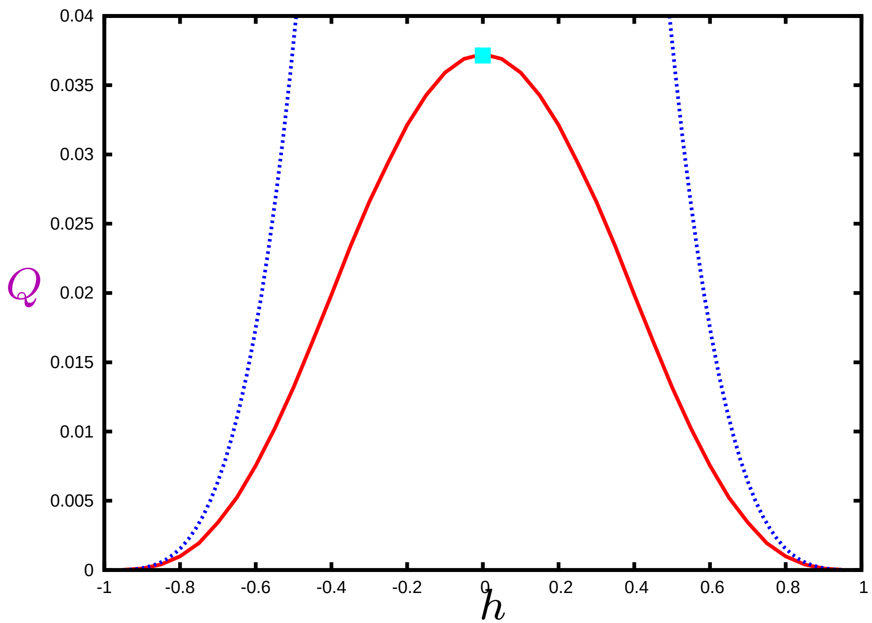
Spreading in a horizontal *circle*

Flat interface at $y = h(x,t)$. Flow in cross-section

$$\nabla^2 u = G + \begin{cases} -\frac{1}{2} & \text{in } h \le y \le 1, \\ +\frac{1}{2} & \text{in } -1 \le y < h \end{cases}$$

and $u = 0$ on $r = 1$. G so no net flux.

Numerical (finite difference 80×80, sor) for flux $Q(h)$ in $h \le y \le 1$.



Thin layer asymptotics, $1 - |h| \ll 1$

No net flow $\Rightarrow$ all pressure gradient in thin layer.

Thin layer $\Rightarrow$ no stress by thick on thin.

Wide compared with thin $\Rightarrow$ 2D, half parabolic profile.

$$Q(h) \sim \frac{32\sqrt{2}}{105}\,(1 - |h|)^{7/2}\,.$$

Symmetric and smooth $Q(h)$, so equally

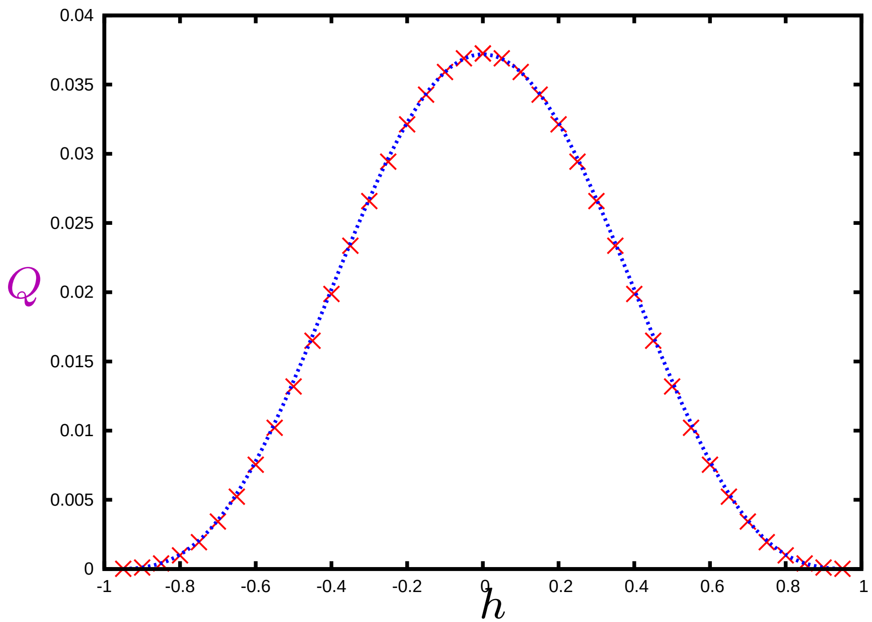
$$Q(h) \sim \frac{4}{105}\left(1 - h^2\right)^{7/2}\,.$$

Lucky break

$$\text{Thin layer asymptotic's} \quad \frac{4}{105} = 0.038095$$
$$\text{50\% fill} \quad Q(0) = 0.037195$$

Hence good approximation (within 1%):

$$Q(h) \approx Q(0)(1-h^2)^{7/2}$$

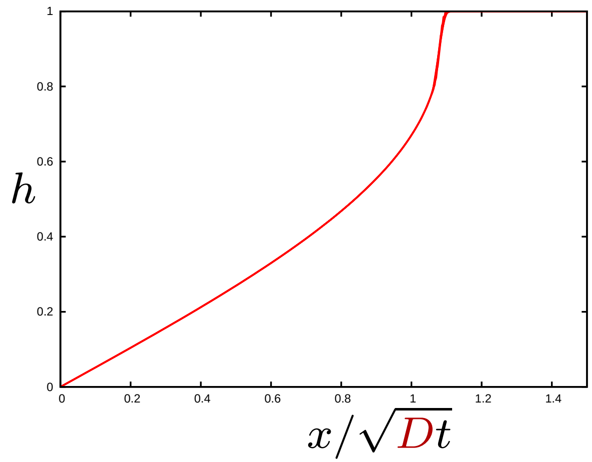


Nonlinear diffusion equation

$$\frac{\partial A(h)}{\partial t} + \frac{\partial Q(h)}{\partial x} = 0,$$

with cross-sectional area $A(h) = 2\cos^{-1} h - h\sqrt{1-h^2}$, so $dA/dh = -2\sqrt{1-h^2}$. Hence

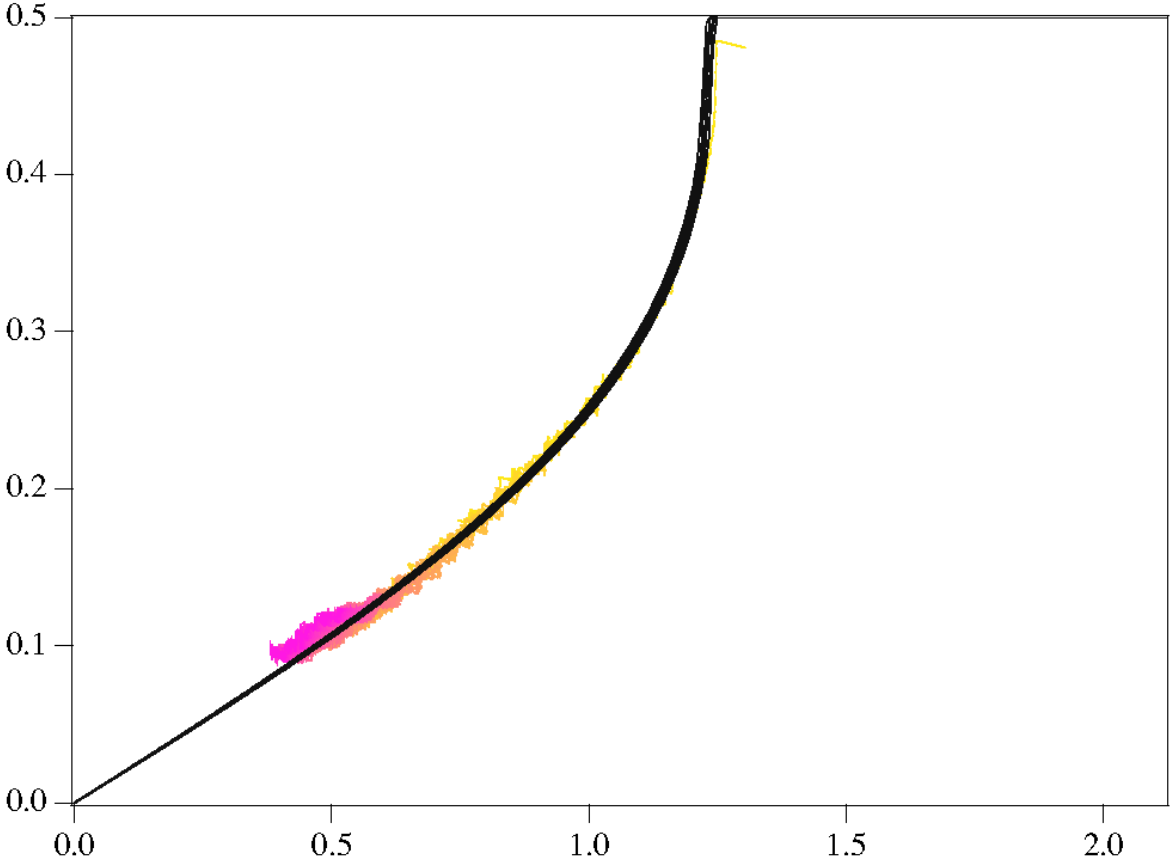
$$\frac{\partial h}{\partial t} = \frac{1}{2\sqrt{1-h^2}} \frac{\partial}{\partial x} \left((1-h^2)^{7/2} \frac{\partial h}{\partial x} \right).$$



$$X_f = \sqrt{Dt}$$

with $D = 0.0108 V_\nu d$,
experiment 0.0108 ± 0.0008.

Comparison with experiment



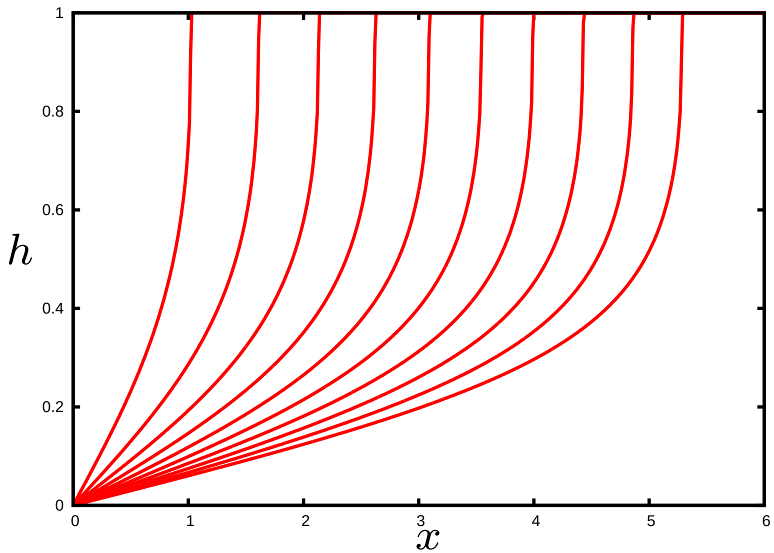
Spreading in nearly horizontal tubes

Flow now driven by difference in pressure gradients in two fluids

$$\Delta\rho g \left(\cos\alpha \frac{\partial h}{\partial x} + \sin\alpha \right).$$

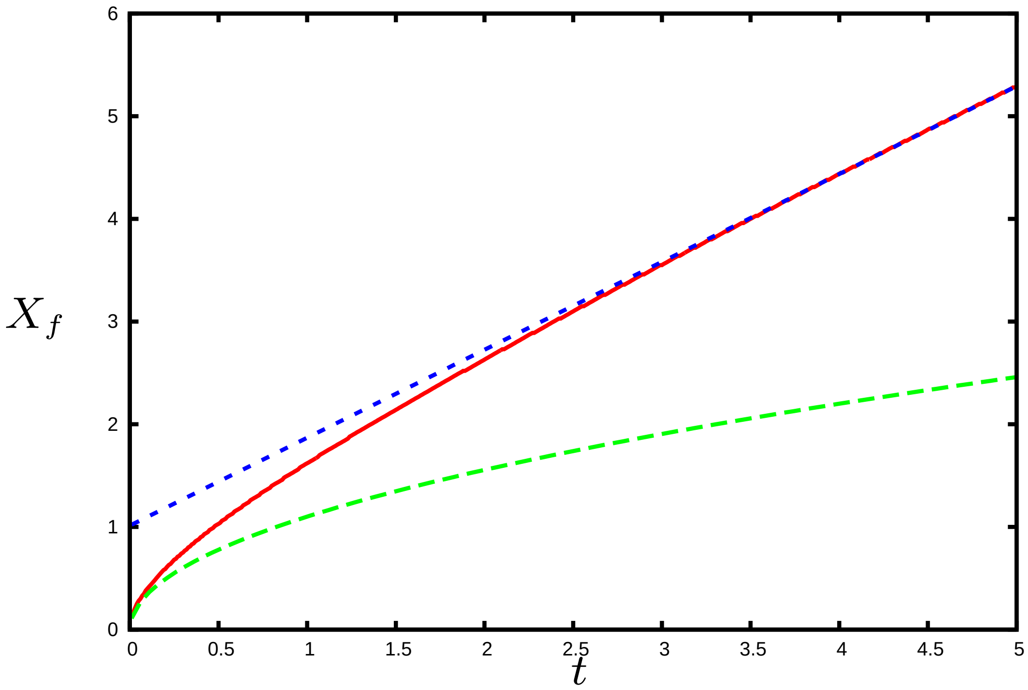
So with suitable rescaling

$$\frac{\partial h}{\partial t} = \frac{1}{2\sqrt{1-h^2}} \frac{\partial}{\partial x} \left((1-h^2)^{7/2} \left(\frac{\partial h}{\partial x} + 1 \right) \right).$$



$t = 0.5, (0.5), 5.0.$

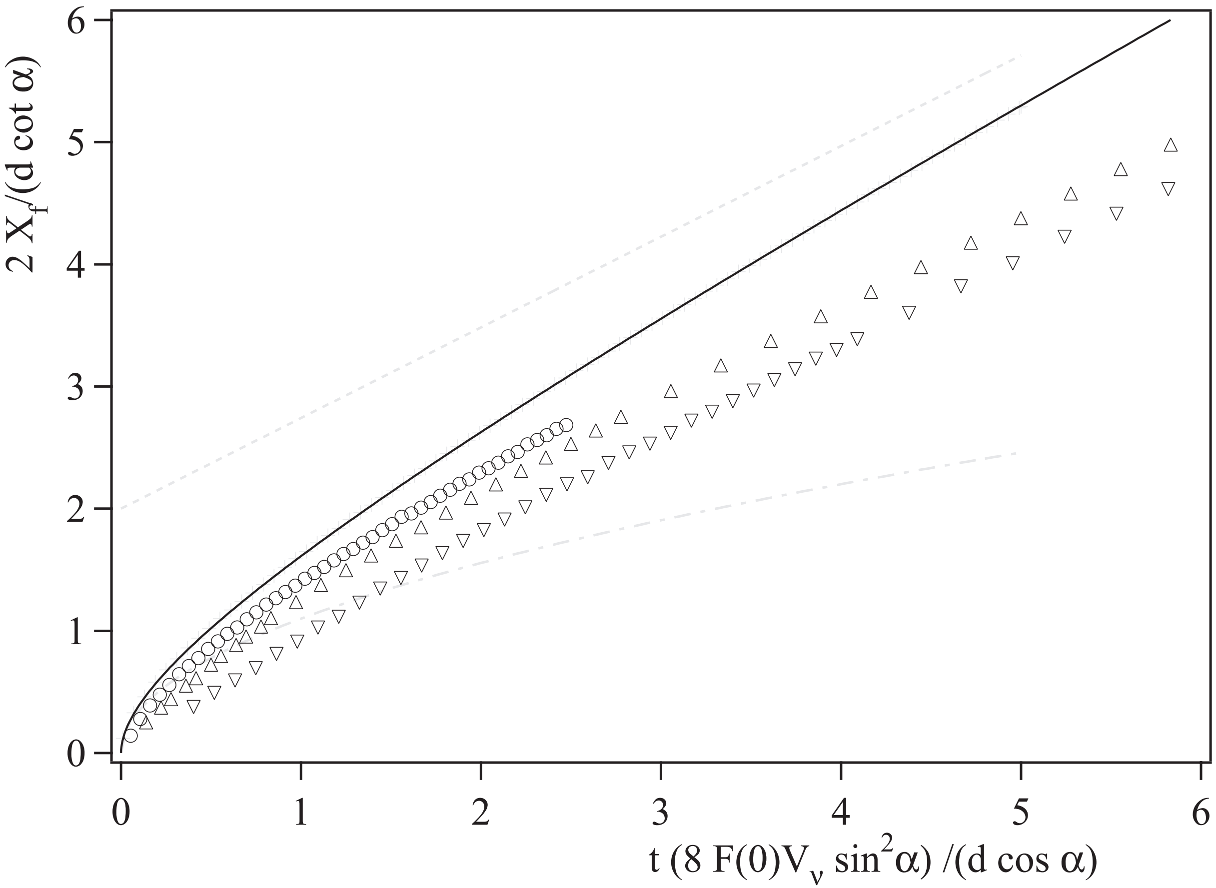
Motion of front $X_f(t)$



Diffusive spreading only at very short times.

Slow to attain long-time velocity.

Experimental motion of front $X_f(t)$



$\alpha = 1^o \circ$,
$2^o \triangle$,
$3^o \nabla$.

Experiments straighter due to different early behaviour – gate release and a short time limited by inertia

Long time approximation?

$$X_f(t) \sim \begin{array}{ll} 0.856t + 1.01 & 4 < t < 5 \\ 0.830t + 1.19 & 7 < t < 11 \\ 0.774t + 1.83 & 20 < t < 25 \\ 0.750t + 2.82 & 90 < t < 100 \\ 0.746t + 3.36 & 145 < t < 150 \end{array}$$

Linear fit fails.

Logarithms at long time

Crude model: horizontal $+$ inclined spreading

$$\dot{X}_f = \frac{V_f L}{X_f} + V_f,$$

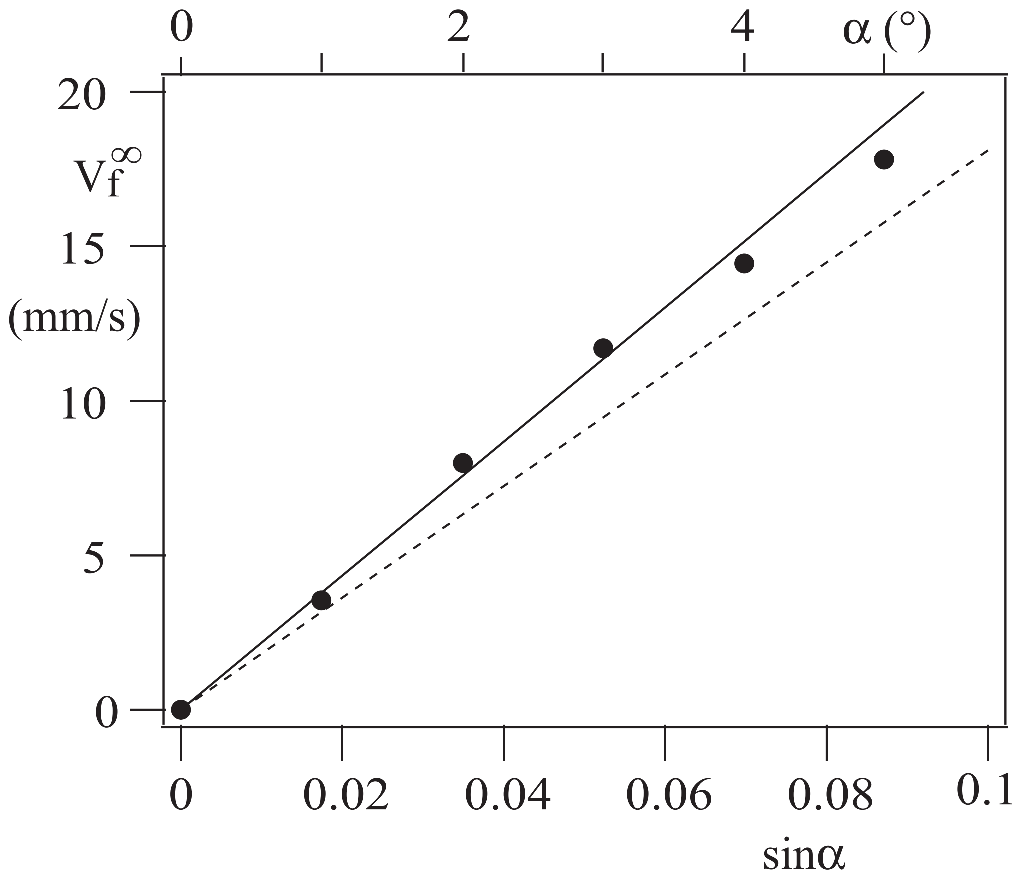
with solution

$$V_f t = X_f - L \log(1 + \frac{X_f}{L}).$$

V_f	L	
0.745	0.798	$0 < t < 5$
0.735	0.851	$0 < t < 20$
0.740	0.812	$0 < t < 50$
0.742	0.791	$0 < t < 100$
0.742	0.789	$0 < t < 200$

Slow logarithmic approach also in experimental data.

Test $V_f = 0.742$



Continuous line is new 0.742 $(0.014 V_\nu \sin\alpha)$,
dashed is old 0.637 $(0.012 V_\nu \sin\alpha)$.

But why $V_f = 0.742$?

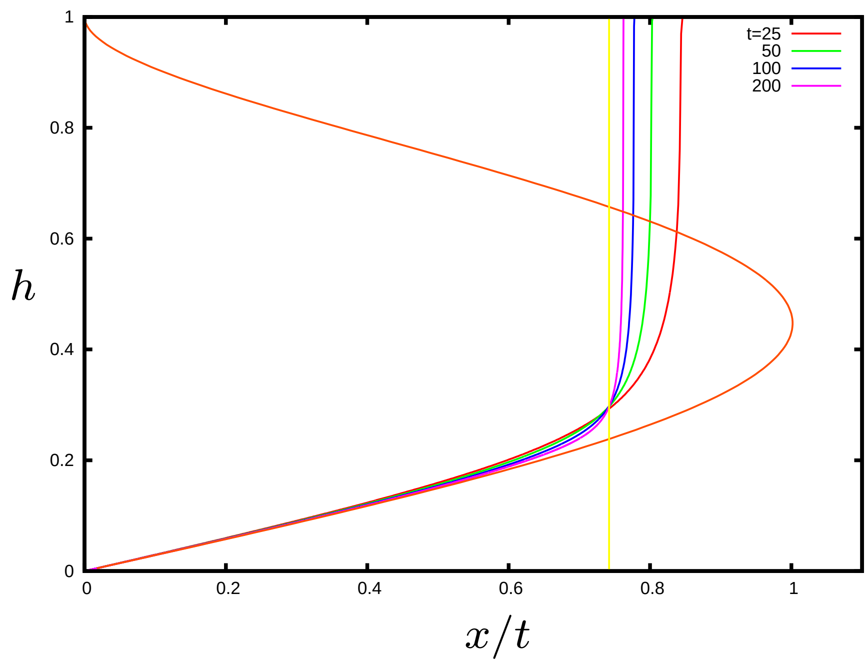
$$\frac{\partial h}{\partial t} = \frac{1}{2\sqrt{1-h^2}} \frac{\partial}{\partial x}\left((1-h^2)^{7/2}\left(\frac{\partial h}{\partial x}+1\right)\right).$$

For $h_x \ll 1$

$$\frac{\partial h}{\partial t} + c(h)\frac{\partial h}{\partial x} = 0$$ with kinematic wave speed $c(h) = \frac{7}{2}h(1-h^2)^2$.

Test rarefaction wave solution by plotting h vs x/t.

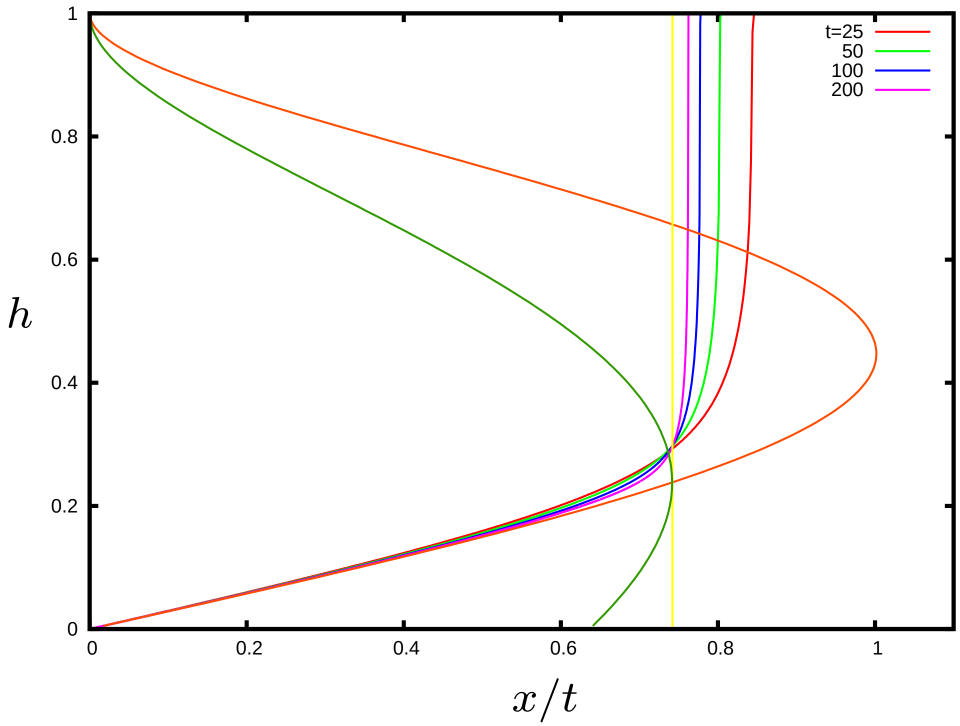
But must shock



Shock waves

Speed of shock $V(h) = \dfrac{Q(h)}{A(h)}$

must equal speed of arriving characteristics $c(h)$.

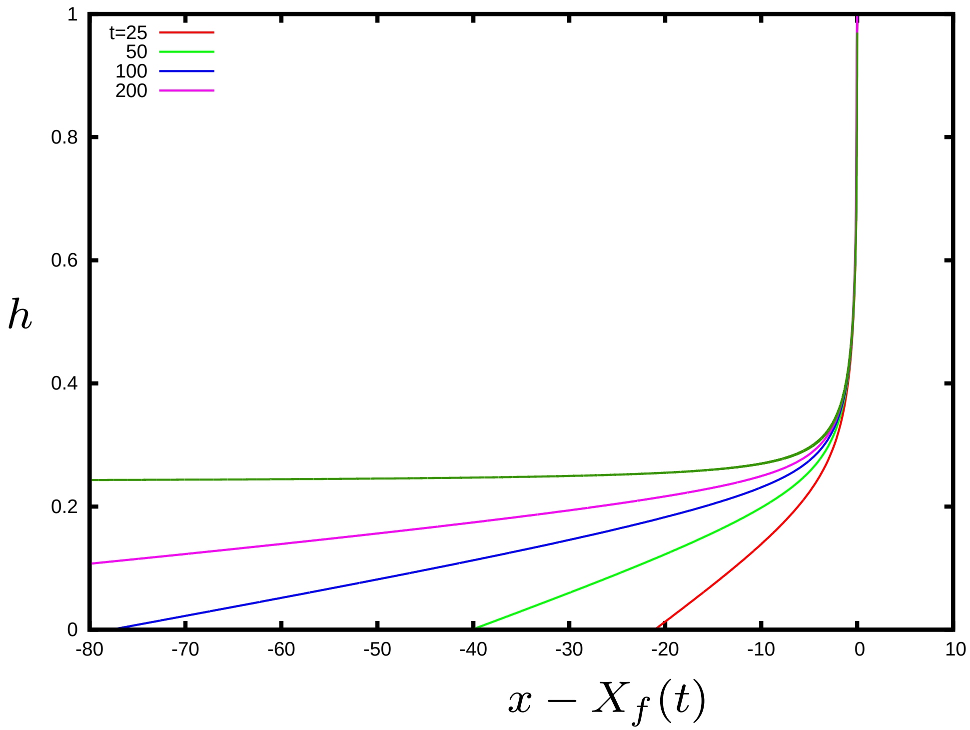


Equal at $h = 0.23817$ with $c = V = 0.74172$.

Form of shock waves

Switch to frame moving with V, then steady form governed by

$$\frac{\partial h}{\partial x} = V\frac{A(h)}{Q(h)} - 1$$



Conclusions

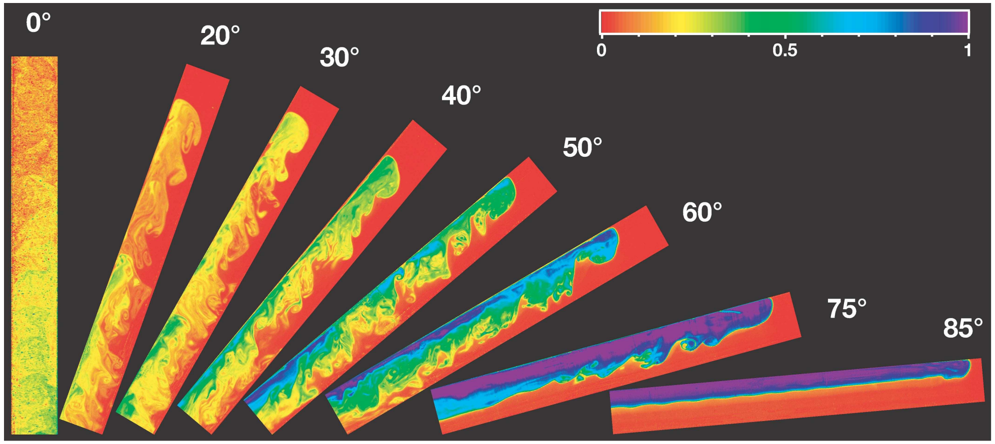
- Front is a shockwave:

  speed selected by matching onto a rarefaction wave.

- Logarithmic approach to long-time:

  value depends on initial condition.

And next



- Vertical : turbulent
- Inclined : nosed controlled gravity current
- Horizontal : viscous counter-current