1 Let $\psi(\mathbf{x})$ be a scalar field and $\mathbf{v}(\mathbf{x})$ a vector field. Show using suffix notation that

$$\operatorname{div}(\psi\mathbf{v}) = (\mathbf{v}.\boldsymbol{\nabla})\psi + \psi \operatorname{div}\mathbf{v} \quad \text{and} \quad \operatorname{curl}(\psi\mathbf{v}) = \operatorname{grad}\psi \times \mathbf{v} + \psi \operatorname{curl}\mathbf{v}.$$

Hence evaluate the divergence and curl of the following:

$$r\mathbf{x}, \quad \mathbf{a}(\mathbf{x}.\mathbf{b}), \quad \mathbf{a}\times\mathbf{x}, \quad \frac{\mathbf{x}-\mathbf{a}}{|\mathbf{x}-\mathbf{a}|^3},$$

where $r = |\mathbf{x}|$ and $\mathbf{a}$ and $\mathbf{b}$ are fixed vectors.

2 Let $\mathbf{u}$ and $\mathbf{v}$ be vector fields. Show using suffix notation that

$$\operatorname{div}(\mathbf{u}\times\mathbf{v}) = \mathbf{v}.\operatorname{curl}\mathbf{u} - \mathbf{u}.\operatorname{curl}\mathbf{v} \quad \text{and} \quad \operatorname{curl}(\mathbf{u}\times\mathbf{v}) = (\mathbf{v}.\boldsymbol{\nabla})\mathbf{u} - \mathbf{v}\operatorname{div}\mathbf{u} + \mathbf{u}\operatorname{div}\mathbf{v} - (\mathbf{u}.\boldsymbol{\nabla})\mathbf{v}.$$

Further show that

$$\operatorname{grad}(\mathbf{u}.\mathbf{v}) = \mathbf{u}\times\operatorname{curl}\mathbf{v} + \mathbf{v}\times\operatorname{curl}\mathbf{u} + (\mathbf{u}.\boldsymbol{\nabla})\mathbf{v} + (\mathbf{v}.\boldsymbol{\nabla})\mathbf{u},$$

so that in particular

$$(\mathbf{u}.\boldsymbol{\nabla})\mathbf{u} = \operatorname{grad}(\tfrac{1}{2}u^2) - \mathbf{u}\times\operatorname{curl}\mathbf{u}.$$

3 Obtain the equation of the plane which is tangent to the surface $z = 3x^2 y \sin(\pi x/2)$ at the point $x = y = 1$.

Take East to be in the direction $(1, 0, 0)$ and North to be $(0, 1, 0)$. In which direction will a marble roll if placed on the surface at $x = 1$, $y = \frac{1}{2}$.

4 A vector field $\mathbf{B}(\mathbf{x})$ is parallel to the normals to a family of surfaces $f(\mathbf{x}) = \text{constant}$. Show that

$$\mathbf{B}.\operatorname{curl}\mathbf{B} = 0.$$

5 Test whether the following force fields are *conservative*, and find a scalar potential if they are.

$$\mathbf{F} = (3x^2yz^2, 2x^3yz, x^3z^2), \qquad \mathbf{G} = (3x^2y^2z, 2x^3yz, x^3y^2),$$

$$\mathbf{H} = \left(3x^2\tan z - y^2\mathrm{e}^{-xy^2}\sin y, (\cos y - 2xy\sin y)\mathrm{e}^{-xy^2}, x^3\sec^2 z\right).$$

6 Test whether the following two-dimensional vector field is *solenoidal*, and find a vector potential in the form $(0, 0, \psi(x, y))$ if it is.

$$\mathbf{u} = ((x\cos y + \cos y - y\sin y)e^x, (-x\sin y - \sin y - y\cos y)e^x).$$

7 Consider

$$\mathbf{A}(\mathbf{x}) = -\int_0^1 \mathbf{x}\times\mathbf{B}(\mathbf{X}=\mathbf{x}t)t\,\mathrm{d}t.$$

Show that $\operatorname{curl}\mathbf{A} = \mathbf{B}$ if $\operatorname{div}\mathbf{B} = 0$ everywhere.

8 Show that the unit basis vectors of cylindrical polar coordinates satisfy

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta \quad \text{and} \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r,$$

all other derivatives of the three basis vectors being zero.

Given that the vector differential operator in cylindrical polars is

$$\boldsymbol{\nabla} = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z},$$

obtain expressions for $\boldsymbol{\nabla}.\mathbf{A}$ and $\boldsymbol{\nabla} \times \mathbf{A}$, where $\mathbf{A} = A_r \mathbf{e}_r + A_\theta \mathbf{e}_\theta + A_z \mathbf{e}_z$.

9 The vector field $\mathbf{B}(\mathbf{x})$ is given in cylindrical polars by

$$\mathbf{B}(\mathbf{x}) = \frac{1}{r}\mathbf{e}_\theta.$$

Evaluate $\boldsymbol{\nabla} \times \mathbf{B}$ when $r \neq 0$ using the formula derived in the previous question. Calculate $\oint_C \mathbf{B}.\mathrm{d}\mathbf{x}$ for C the circle $r = 1$, $0 \leq \theta \leq 2\pi$ and $z = 0$. Does Stokes' theorem apply? Why not?

10 By applying the divergence theorem to the vector field $\mathbf{k} \times \mathbf{B}$, where $\mathbf{k}$ is an arbitrary constant vector and $\mathbf{B}(\mathbf{x})$ is a vector field, show that

$$\int_V \boldsymbol{\nabla} \times \mathbf{B}\, \mathrm{d}V = -\int_A \mathbf{B} \times \mathrm{d}\mathbf{A},$$

where the surface A encloses the volume V.

Verify this result when A is the sphere $|\mathbf{x}| = R$ and $\mathbf{B} = (z, 0, 0)$ in Cartesian coordinates.

11 By applying Stokes' theorem to the vector field $\mathbf{k} \times \mathbf{B}$, where $\mathbf{k}$ is an arbitrary constant vector and $\mathbf{B}(\mathbf{x})$ is a vector field, show that

$$\oint_C \mathrm{d}\mathbf{x} \times \mathbf{B} = \int_A (\mathrm{d}\mathbf{A} \times \boldsymbol{\nabla}) \times \mathbf{B},$$

where the curve C bounds the open surface A.

Verify this result when C is the unit square in the x-y plane with opposite vertices at $(0, 0, 0)$ and $(1, 1, 0)$ and $\mathbf{B} = \mathbf{x}$.

12 The open surface A, which has a unit normal $\mathbf{n}$, is bounded by the curve C. The vector field satisfies $\mathbf{v}.\mathbf{n} = 0$ on the surface A. The vector field $\mathbf{m}(\mathbf{x})$ is a unit vector everywhere and satisfies $\mathbf{m} = \mathbf{n}$ on the surface A. By applying Stokes theorem to $\mathbf{m} \times \mathbf{v}$, show that

$$\int_A (\delta_{ij} - n_i n_j)\frac{\partial v_i}{\partial x_j}\, \mathrm{d}A = -\oint_C \mathbf{v}.\mathbf{n} \times \mathrm{d}\mathbf{x}.$$

[Hint: Q2 and $(\mathbf{m}.\mathbf{m})' = 0$*?]*

I would appreciate any comments and corrections from students and supervisors. Please e-mail `ejh1@cam.ac.uk`.