MATHEMATICAL TRIPOS PART IIB — Prof. E.J. Hinch
'Waves in Fluid and Solid Media' — January 2003

Example Sheet 1: Sound Waves‡

1. *Propagating & decaying (evanescent) waves.* Seek a z-independent solution to the wave equation of the form

$$\phi = A\exp(i\alpha x - i\omega t) f(y) ,$$

where ϕ is the velocity potential, $\alpha, \omega > 0$, and $\alpha > \omega/c_0$. Hence find an acceptable solution in $y \geq 0$ if there is no disturbance as $y \to \infty$, and

$$v = \exp(i\alpha x - i\omega t) \quad \text{on} \quad y = 0 ; \qquad \mathbf{u} = (u, v, 0) .$$

Indicate the *approximate* width of the layer (next to the surface $y = 0$) to which the waves are confined.

From consideration of the acoustic energy flux, verify:
(a) that no energy is propagated perpendicular to the wall;
(b) that at any y-station, the time averaged energy flux parallel to the wall, i.e. $< I_x >$, satisfies

$$< I_x > = < E > c ,$$

where the phase velocity along Ox is $c = \omega/\alpha < c_0$ (note that the disturbance and its energy travel subsonically along the surface with respect to the sound speed in $y > 0$).

Comment on whether *evanescent* waves could arise (i) in a uniform, doubly infinite, medium, $-\infty < y < \infty$, and (ii) in a bounded medium $0 < y < y_{max}$?

2. *Spherical waves.* The wave equation with spherical symmetry is

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\tilde{p}) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\tilde{p} = 0 .$$

(a) Show that the general solution is

$$\tilde{p} = \frac{f(t - r/c_0)}{4\pi r} + \frac{g(t + r/c_0)}{4\pi r} , \tag{1}$$

where f and g are arbitrary functions. The first and second parts of the solution represent outgoing and incoming waves respectively: why?

(b) Calculate the radial velocity for an outgoing wave. Show by considering the volume outflow from an infinitesimally small sphere centred on $r = 0$, that $f(t) = \dot{q}(t)$, where $q(t)$ is the mass outflow from $r = 0$.

Definition: $\dot{q}(t)$ is the *strength of the monopole field.*

(c)* From consideration of $\lim_{V\to 0} \int_V \Box^2 \tilde{p}\, dV$ for small volumes centred on appropriate points in space, deduce that the governing equation for a monopole is in fact

$$\Box^2 \tilde{p} = -\dot{q}(t)\delta(\mathbf{x}) ,$$

where δ is the Dirac delta function.

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‡ Corrections and suggestions should be emailed to `E.J.Hinch@damtp.cam.ac.uk`.

3. *Harmonic series.* Suppose that (1) describes a possible small acoustic disturbance of an ideal fluid in a conical tube of any cross-sectional shape.‡ Model an oboe or saxophone (with all the finger-holes closed) as a conical tube of length ℓ and of small angle, at one end of which the cross-sectional area is effectively zero and $\tilde{p}$ finite, and at the other end of which $\tilde{p}$ may be assumed zero.† Show that the instrument has a set of normal modes of frequencies

$$\omega\ell/c_0 = n\pi \quad (n \text{ an integer}) . \tag{2}$$

If, instead, the larger end is closed, so that the particle velocity or displacement is zero there, show that the corresponding normal-mode frequencies are the solutions of

$$\omega\ell/c_0 = \tan(\omega\ell/c_0) , \tag{3}$$

and find their high-frequency limit. [The set of frequencies (2) forms a musical 'harmonic series', while the set (3) does not.]

4. *Radiated power.* (*a*) A simple source at the origin 0 creates mass at a rate $q(t)$ per unit time (see question 2). Show that in the *far-field*, $r \gg c_0/\omega$, (i) the kinetic energy density, E_k, (ii) the potential energy density, E_p, and (iii) the wave-energy flux, $\mathbf{I} = \tilde{p}\mathbf{u}$, of the outgoing spherical wave *approximately* satisfy plane-wave relations (assume that the 'source strength', i.e. the time derivative $\dot{q}(t)$, is of order ωq for some constant ω). Hence show that the total power radiated across a large sphere of radius R is

$$\left(\dot{q}(t - c_0^{-1}R)\right)^2 /4\pi\rho_0 c_0 . \tag{4}$$

(*b*) Show that as $r \to 0$,

$$E_k/E_p \to (c_0 q/\dot{q}r)^2 = O(c_0/r\omega)^2 \quad \to \infty \quad \text{as} \quad r \to 0 ,$$

and comment on this result.

(*c*) Suppose now that *plane* sound waves are generated in the half-space $x > 0$ by a piston which causes mass to flow back and forth across the plane $x = 0$ at a rate $Q(t)$ per unit area. Show that the power radiated across area A of a plane $x = X(> 0)$ is

$$\left(q(t - c_0^{-1}X)\right)^2 c_0/\rho_0 A , \tag{5}$$

where $q = AQ$.

(*d*) If $q(t) = \mathrm{Re}(\bar{q}e^{-i\omega t})$, deduce from (4) and (5) that if the same mass source were placed at the closed end of a semi-infinite narrow tube of cross-sectional area $A \ll 4\pi^2 c_0^2/\omega^2$, then the time-averaged power radiated would increase by a large factor approximately equal to $4\pi c_0^2/A\omega^2$ (assume one-dimensional motion in the tube, and ignore details of the flow very near the source). This is one of the principles behind the 'horn loudspeaker'.

(*e*)* Using the method of images, calculate the time-averaged power radiated by a point source placed well within a wavelength (i) of a plane rigid boundary, and (ii) of the corner between two plane rigid boundaries at right-angles. Will a whistle sound louder if blown near a wall?

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‡ Since sound waves are longitudinal waves, the boundary condition, that the velocity component normal to the wall vanishes, is satisfied.

† In reality this is a good approximation only if the radius of this larger end is much less than c_0/ω.

5. *An oscillating bubble* (Old Tripos: 93124). A spherical bubble makes small spherically symmetric oscillations in an ideal fluid. You may assume that the internal dynamics of the bubble produces a pressure in the fluid on the bubble surface $-\kappa(a-a_0)$, where the radius $a(t)$ makes *small* oscillations about the mean value a_0. By applying linearised velocity and pressure boundary conditions at $a = a_0$, derive the equation of motion for the oscillations

$$\rho_0 a_0 \ddot{a} + \frac{\kappa a_0}{c_0}\dot{a} + \kappa(a - a_0) = 0 \ ,$$

where ρ_0 is the undisturbed density of the fluid and c_0 is the sound speed (you may quote results from question 2). What is the mechanism of energy loss represented by the 'damping' ($\dot{a}$ term) in the equation?

6. *Traffic flow.* Let $\rho(x,t)$ denote the density of cars per unit length of road, and let $q(x,t)$ denote the numbers of cars passing position x per unit time (in our ideal world there are no lorries or vans). Postulate that $q = Q(\rho)$, i.e. that the traffic flow is a known function of the local density of traffic. From conservation of cars deduce that

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0 \ ,$$

where $c(\rho) = Q'(\rho)$. Let $V(x,t) = Q/\rho$ be the local car velocity. If V is assumed to be a monotonic decreasing function of ρ, comment on the relative magnitudes of the propagation velocity and car velocity (does this agree with your experience?).

7. *Shock formation..* At a certain instant the velocity u in a one-dimensional 'simple wave' of finite amplitude, propagating through a perfect gas, has the x-dependence

$$u = u_m \sin kx \ ,$$

where u_m and k are constants. The wave subsequently propagates in the positive x direction. Show that a single *wavecrest* (i.e. a local maximum of $u(x,t)$) travels a distance

$$\frac{1}{k}\left(\frac{2c_0}{(\gamma+1)u_m} + 1\right) \tag{6}$$

before shock waves form.

Comment: When $k = 2\pi \times (1\text{kHz})/c_0$, $c_0 = 340\text{ms}^{-1}$, $\gamma = 1.4$, and $u_m = 0.05\text{ms}^{-1}$ (equivalent to 120dB, i.e. the threshold of pain for the ear), the distance (6) is about 320m. At 100dB we get 3200m — but dissipation then becomes enough, under ordinary conditions, to invalidate the assumption of an ideal fluid.