MATHEMATICAL TRIPOS PART IIB — Prof. E.J. Hinch
'Waves in Fluid and Solid Media' — February 2003

Example Sheet 2: Shocks and Elastic Waves

1. *An expansion fan.* A piston confines an ideal, compressible fluid to the right-hand half, $x > 0$, of an infinite tube. The piston and the fluid are initially at rest for times $t < 0$. At time $t = 0$ the piston starts moving with constant speed V *away* from the fluid. Show that there is just one region in the (x, t) plane, R_2 say, throughout which the local sound speed c takes a constant value c_2 different from its value c_0 in the undisturbed region, R_0 say. Show by *reductio ad absurdum*, or otherwise, that all the C_+ characteristics lying outside both R_2 and R_0 must pass through the origin, and deduce that the form of the disturbance at time $t > 0$ is

$$u + c = \left\{ \begin{array}{ll} c_2 - V\ , & -Vt \le x \le (c_2 - V)t \\ xt^{-1}\ , & (c_2 - V)t \le x \le c_0 t \\ c_0\ , & x \ge c_0 t \end{array} \right\} . \qquad (7)$$

Sketch a graph of this *expansion fan* as a function of x. Would the individual graphs of u and c look the same as the graph for $(u + c)$: $(a)^*$ for a fluid with $p \propto (e^\rho - 1)$, and (b) for a perfect gas with $p \propto \rho^\gamma$?

For the case of a perfect gas explain why the solution (7) is physically meaningless when

$$V > 2(\gamma - 1)^{-1} c_0\ .$$

Deduce that a gas expanding one-dimensionally into a vacuum does so with the *escape speed*

$$2(\gamma - 1)^{-1} c_0\ .$$

2. *Two expansion fans* (Old Tripos: 75425). A perfect gas, with constant specific heats in the ratio γ, is initially at rest with uniform sound speed c_0. It is confined by two pistons to the region $0 < x < 2\ell$ of a long cylindrical tube. At time $t = 0$, both pistons are set into impulsive motion away from the gas with constant velocities $u = -V < 0$ and $u = U > 0$.

(i) For $0 \le t \le \ell/c_0$ show that in the part $x \le \ell$ of the tube (which cannot have been reached by any signal from the piston initially at $x = 2\ell$), every C_+ characteristic is a straight line.† Show that the fluid velocity u takes the value

$$u = \frac{2}{\gamma + 1}\left(\frac{x}{t} - c_0\right) \quad \text{for} \quad \left(c_0 - \frac{\gamma + 1}{2}V\right)t < x < c_0 t\ .$$

Give the corresponding value of c. Find the shape of the C_- characteristics when u and c take these values.

(ii) Deduce that, when $t > \ell/c_0$, the equation

$$u = \frac{2}{\gamma + 1}\left(\frac{x}{t} - c_0\right)$$

† Results derived in question 1 should be quoted.

is satisfied only in the smaller interval

$$\left(c_0 - \frac{\gamma+1}{2}V\right)t < x < \frac{\ell}{\gamma-1}\left((\gamma+1)\left(\frac{c_0 t}{\ell}\right)^{(3-\gamma)/(\gamma+1)} - 2\left(\frac{c_0 t}{\ell}\right)\right) .$$

(iii) For a case with V/c_0 about $\frac{1}{2}$ and U/c_0 about $\frac{1}{4}$, give a rough sketch indicating **four** areas of the (x,t) plane throughout each of which u takes a different constant value, to be specified.

3. *A piston-generated shock.* A piston moves with constant positive velocity u_1 into a perfect gas of specific heat ratio $\gamma > 1$, generating a shock wave which moves ahead of the piston. Show that a possible solution of all the relevant equations is one in which the gas is at rest beyond the shock, at pressure p_0, and is moving with constant velocity u_1 in the region between the piston and the shock, throughout which region the density and pressure also take constant values ρ_1, p_1 which are determined by

$$\frac{\rho_1}{\rho_0} = \frac{2\gamma + (\gamma+1)\beta}{2\gamma + (\gamma-1)\beta} , \qquad \frac{1}{\beta^2} + \frac{\gamma+1}{2\gamma\beta} = \frac{c_0^2}{\gamma^2 u_1^2} ,$$

where β is the shock strength defined as $(p_1 - p_0/p_0 > 0$, and ρ_0 and c_0 are the density and sound speed of the undisturbed gas. The internal energy per unit mass of a perfect gas of density ρ at pressure p equals $p/(\gamma-1)\rho$.

Hint. In order to demonstrate that the propagation speed C of the shock does indeed exceed the given velocity u_1, it *may* be helpful to show that

$$C = c_0 \left(1 + \frac{\gamma+1}{2\gamma}\beta\right)^{1/2} .$$

4.* *A method to generate shock waves in a 'shock tube'* (Old Tripos: 85427). An infinitely long uniform tube contains two perfect gases separated by a membrane at $x = 0$. The gas in $x > 0$ has pressure p_1, density ρ_1 and specific heat ratio γ_1; the corresponding values for the gas in $x < 0$ are p_2, ρ_2, γ_2 where $p_2 > p_1$. At $t = 0$ the membrane is burst. Assuming that the interface between the two gases remains plane and moves with constant speed V, use the one-dimensional equations of motion to show that there are three regions in the tube in which the pressure is uniform,

$$p = p_2 \quad \text{for} \quad x < -\left(\frac{\gamma_2 p_2}{\rho_2}\right)^{1/2} t ,$$

$$p = p_1 \quad \text{for} \quad x > Ut ,$$

$$p = p_m \quad \text{for} \quad -\left[\left(\frac{\gamma_2 p_2}{\rho_2}\right)^{1/2} - \frac{\gamma_2+1}{2}V\right] t < x < Ut ,$$

where p_m is as yet unknown, and the shock velocity, U, is a constant to be found in terms of p_m, p_1, ρ_1, γ_1.

Show that V is related to p_m by the following two equations:

$$V = (p_m - p_1)\left(\tfrac{1}{2}\rho_1\left[(\gamma_1+1)p_m + (\gamma_1-1)p_1\right]\right)^{-1/2} ,$$

$$V = \frac{2}{\gamma_2 - 1}\left(\frac{\gamma_2 p_2}{\rho_2}\right)^{1/2}\left[1 - \left(\frac{p_m}{p_2}\right)^{(\gamma_2-1)/2\gamma_2}\right] ,$$

and hence show that there is a unique solution for p_m and V.

5. *Generalisation of Hooke's law.* For an uniaxial tension,

$$\sigma_{11} = \sigma\,, \quad \sigma_{ij} = 0 \quad \text{for} \quad i \neq 1,\ j \neq 1\,,$$

show that $E\mathbf{u} = \sigma(x, -\nu y, -\nu z)$, where the Young's modulus E, and Poisson's ratio ν, should be expressed in terms of the bulk modulus and shear modulus. For what range of values of E and ν is the elastic energy density positive definite? [In practice $\nu > 0$, so that an axial strain induces a radial contraction.]

Typical values of E (in units of $10^{10}\,\mathrm{Nm}^{-2}$) and ν for materials at 20° are:

	E	ν		E	ν
Aluminium	7.0	0.35	Soft iron	12.1	0.29
Copper	13.0	0.34	Mild steel	21.2	0.29

A cylindrical steel wire of unstrained radius 1 mm and length 1 m has a weight of 1 kg suspended from it. Find the increase in length, and the decrease in diameter, of the wire. [Very high forces generate very small strains — hence in engineering practice linear theory is very good. Beyond modest strains, e.g. 10^{-2}, metals turn plastic, and the engineers normally redesign their structures.]

6. *A hanging rod.* [Optional, since the question has little to do with waves.] A circular homogeneous cylinder of length l, radius a and density ρ hangs under its own weight. Verify that, under suitable boundary conditions (to be specified), an appropriate displacement field is

$$\mathbf{u} = \frac{\rho g}{E}\left(\nu x(z-l),\ \nu y(z-l),\ lz - \tfrac{1}{2}z^2 - \tfrac{1}{2}\nu(x^2+y^2)\right)\,,$$

where E and ν are the Young's modulus and Poisson's ratio for the solid. Sketch the deformed shape, and estimate the size of the effect for an iron beam. How long must the beam be for yield to occur?

7. *Energy fluxes.* Find the energy flux vectors both for a plane harmonic P-wave, and for a plane harmonic S-wave, travelling on its own in homogeneous material. Find also the energy flux for the case where both waves travel together in the same direction. Can the separate energy flux *vectors* be added to give the flux in the second case?

8. *Reflection of a SV-wave.* A solid with elastic wavespeeds c_P, c_S occupies the region $y < 0$ and is bonded to a rigid barrier $y = 0$. A SV-wave with elastodynamic potential

$$\psi = Ae^{i(\omega t - kx\sin\theta - ky\cos\theta)}\mathbf{e_z}\,,$$

is incident on the barrier. If $\sin\theta < q = c_S/c_P$, obtain the reflected waves. If $\sin\theta > q$ show that a solution may be found consisting of a reflected SV-wave together with a compressional interface wave near $y = 0$.