MATHEMATICAL TRIPOS PART IIB — Prof E.J. Hinch
'Waves in Fluid and Solid Media' — February 2003

Example Sheet 3: Dispersive Waves

1. *Stoneley waves.* Examine the propagation of surface waves (whose amplitudes decay away from the interface in both directions) at the interface between a homogeneous elastic solid and an elastic fluid. Note: since the fluid can support only compressional waves, continuity of tangential displacement cannot be imposed at the interface (i.e. the fluid can slip past the solid).

With a fluid density $\bar{\rho}$ and a fluid sound speed $\bar{c}$ you should find the analogue of Rayleigh's equation as

$$\frac{c^4}{c_S^4}\frac{\bar{\rho}}{\rho}\left(\frac{1-c^2/c_P^2}{1-c^2/\bar{c}^2}\right)^{1/2} = 4\left(1-c^2/c_P^2\right)^{1/2}\left(1-c^2/c_S^2\right)^{1/2} - \left(2-c^2/c_S^2\right)^2 .$$

* Discuss solutions to this equation.

2. *Love waves under a rigid surface.* A layer of thickness h has a rigid upper surface, a shear modulus $\bar{\mu}$ and a shear wavespeed $\bar{c}_S$. It overlies a uniform half space with shear modulus μ and shear wavespeed c_S ($c_S > \bar{c}_S$). Find the dispersion relation for Love waves of frequency ω and wavenumber k to exist on this structure. Determine the cut-off frequency for each mode, and the limiting phase velocity for high-frequency propagation. Sketch the phase velocity curve, *and group velocity curve*, as a function of frequency.

3. *SH waves in an elastic layer.* Consider the propagation of SH waves in a layer made from a solid with shear modulus μ and shear wavespeed c_S. Assume that the layer is of thickness h, and that the boundaries at $y = 0$ and $y = h$ are free surfaces. Derive the dispersion relation for both symmetric and antisymmetric modes. Verify that the time averaged wave energy flux is equal to the time averaged wave energy density times the group velocity c_g.

4. *The Klein–Gordon Equation.* The motion of a 2D membrane supported by springs and subject to a forcing $f(x,t)$ per unit length, is governed by the Klein–Gordon equation

$$m\frac{\partial^2\eta}{\partial t^2} - T\frac{\partial^2\eta}{\partial x^2} + K\eta = f .$$

Show that

$$\frac{d}{dt}\int_{x_1}^{x_2}\left(\tfrac{1}{2}m\dot{\eta}^2 + \tfrac{1}{2}T\eta_x^2 + \tfrac{1}{2}K\eta^2\right)dx = \int_{x_1}^{x_2} f\dot{\eta}\,dx + F(x_1,t) - F(x_2,t) ,$$

where

$$F(x,t) = -T\dot{\eta}\eta_x .$$

Give a physical interpretation to each term. Show that in the case of an unforced membrane, the time averaged wave energy flux is equal to the time averaged wave energy density times the group velocity c_g.

5. *Properties of finite-depth capillary-gravity waves.* For waves of radian wave-number k on water of density ρ and uniform depth h, taking surface tension T as well as the gravitational acceleration g into account, the dispersion relation is

$$\omega^2 = k(g + Tk^2/\rho)\tanh\ kh\ .$$

Show that for sufficiently large k the group and phase velocities c_g and c_p become proportional to $k^{\frac{1}{2}}$ and independent of g and h, and that $c_g \sim \frac{3}{2}c_p$.

In ripple-tank experiments it is desired to keep c_g and c_p as nearly constant as possible for smaller values of kh. By expanding ω^2 in ascending powers of k, determine approximately what value of h, h_0 say, should be used. Show also that for $h > h_0$ there must exist a minimum value of the group velocity at some finite non-zero value of k.

Comment: For water a typical value of $T/\rho g$ would be $7.5\,\mathrm{mm}^2$, so that $h_0 = 4.74\,\mathrm{mm}$.

6. *An example of a causal solution where the wavecrests move* **toward** *the source.* What is meant by the 'radiation condition'? Show that the solution of

$$\frac{\partial^4\psi}{\partial x^2 t^2} - \alpha^2\psi = 0\ ,$$

which represents steady propagation into $0 < x < \infty$ of waves generated by a time-harmonic boundary condition

$$\psi|_{x=0} = ae^{-i\omega t}\ ,$$

is

$$\psi = ae^{-i\omega(t+(\alpha x/\omega^2))} \quad (\alpha > 0) \quad [\text{sic}]\ .$$

[One physical system to which this problem corresponds is that of a vertical tube (x vertical), containing a density-stratified fluid; this acts as a waveguide for internal gravity waves whose wavelength $2\pi/k$ is short compared with the dimensions of the tube.]

7. *An example of stationary phase.* (Old Tripos: 77126). A hypothetical physical system permits one-dimensional wave propagation in the x-direction according to an equation of the form

$$\frac{\partial\psi}{\partial t} - \beta\frac{\partial^3\psi}{\partial x^3} = 0 \quad (\beta > 0, \quad \text{constant})\ . \qquad (*)$$

Write down the corresponding dispersion relation and sketch graphs of frequency, phase velocity and group velocity as functions of wave number. Determine whether it is the shortest or longest waves which are to be found at the front of a dispersing wave train arising from a localised initial disturbance. Do the wave crests move faster or slower than the wave train as a whole?

An initial disturbance is given in the form of a Fourier integral,

$$\psi(x, 0) = \int_{-\infty}^{\infty} A(k)e^{ikx}\,dk\ .$$

Write down the corresponding solution of (*) and use the method of stationary phase to obtain an approximation to this solution for large t with t/x held constant. Do not consider complex k, nor give rigorous estimates of error. You may assume that

$$\int_{-\infty}^{\infty} e^{\pm iu^2}\,du = \pi^{1/2}e^{\pm i\pi/4}\ .$$

8. *The generation of gravity waves.* (i) Find a combination of gravity waves on deep water travelling in the directions x increasing and x decreasing that satisfies the conditions

$$\zeta = \zeta_0 \cos kx, \quad \partial\zeta/\partial t = 0 \, ,$$

at time $t = 0$, where ζ is the upward displacement of the free surface at $y = 0$ (assume *no* surface tension).

(ii) In a deep and very long channel, parallel to the x-axis, the water surface is distorted by the action of air jets into a shape

$$\zeta = -\zeta_0 e^{-(x/a)^2} \, ,$$

independent of z, and then released from rest at time $t = 0$. Obtain the subsequent shape of the free surface as a sum of two Fourier integrals. Use the method of stationary phase to obtain their approximate value when x and t are both large and positive.

9*. *A model system demonstrating the ideas of group velocity.* A very large number of identical pendulums are arranged with their points of support equally spaced along a long horizontal line so that they can all swing in the same vertical plane. The bob of each pendulum has mass m and is at a distance ℓ from its point of support. Every adjacent pair of bobs is a distance h apart and is joined by a light spring, which resists any change in this distance of separation with a force equal to K times the change in distance, where $K \gg mg/\ell$. Show that sinusoidal wave motions with wavelength λ much greater than h can propagate along the row of pendulums, provided that the radian frequency ω satisfies

$$\omega = \pm\sqrt{\left(\frac{g}{\ell} + \frac{Kh^2k^2}{m}\right)}$$

where $k = 2\pi/\lambda$ is the (radian) wavenumber. Use the idea of group velocity to give a general description (without doing detailed calculations) of what would happen if from time $t = 0$ onwards the pendulum at one extreme end were forcibly given a regular sinusoidal displacement with radian frequency $1.5\sqrt{(g/\ell)}$.

A precisely analogous device (but using transverse instead of longitudinal vibrations) was frequently used by Osborne Reynolds to demonstrate the phenomenon of group velocity; see Lighthill, p. 281, ex. 6 for further details. Note that the acoustic waveguide has a dispersion relation of the same form, $\omega^2 = a^2 + b^2k^2$ (a, b real constants). Yet another physical system with this form of dispersion relation is an ionised gas through which plane electromagnetic waves propagate (Feynman Lectures II, §7.3). The electrons play a role analogous to the pendulum bobs, the comparatively immobile positive ions corresponding to the points of support, and the net Coulomb force on the electrons to the gravitational restoring force. (Frequencies well above the ionospheric cutoff frequency a must be used for satellite communications!)