MATHEMATICAL TRIPOS PART IIB — Prof E.J. Hinch
'Waves in Fluid and Solid Media' — March 2003

Example Sheet 4: Ray Theory

1. *The governing equation of sound waves in a stratified fluid.* A gas exactly satisfies the equation of state

$$p(\rho, T) = \lambda \rho^{\frac{1}{2}} ,$$

where λ is a constant independent of the temperature T. The gas fills $z \geq 0$ and is at rest in a gravity field $\mathbf{g} = (0, 0, -g)$. Show that the density is given by

$$\rho = \frac{\lambda^2}{g^2(z - z_0)^2} ,$$

where $z_0 < 0$ is a constant.

Show that linear sound waves propagating in the above gas satisfy

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} - \nabla . \left(c_0^2 \nabla \tilde{\rho} - \tfrac{3}{2} \tilde{\rho} \mathbf{g} \right) = \frac{\partial^2 \tilde{\rho}}{\partial t^2} - \nabla . \left(\rho_0^{3/2} c_0^2 \, \nabla \left(\rho_0^{-3/2} \tilde{\rho} \right) \right) = 0 ,$$

where $\tilde{\rho}(\mathbf{x}, t)$ is the density perturbation, $\rho_0(z)$ is the unperturbed density, and $c_0(z)$ is the local speed of sound. Deduce that a slowly varying approximation for sound waves with wavenumber k is valid in $z \geq 0$ if $|kz_0| \gg 1$.

2. *The wave-crest pattern near a shore line.* (Old Tripos: 87327). Surface waves on water have a dispersion relation $\omega = \Omega(\kappa; x, z)$ where $\kappa^2 = k_1^2 + k_3^2$, (x, z) are coordinates in the plane of the surface, and the medium is 'slowly varying' in the (x, z) coordinates. Assume the relation

$$\omega_t + (\mathbf{c}_g . \nabla)\omega = 0 ,$$

where $\mathbf{c}_g$ is the group velocity, and deduce
(*a*) that ω is constant on rays, $dz/dx = k_3/k_1$,
(*b*) that the wave crests at any instant are given by $dz/dx = -k_1/k_3$.

Surface tension effects are negligible, and the wave motion takes place over a sloping beach of depth $h(x) = \alpha x^{1/2}$, with α a small positive constant. The dispersion relation for such waves may be assumed to be given by $\Omega^2 = g\kappa \tanh \kappa h$. Far from the shore-line $x = 0$, the waves are plane, have frequency ω, and have angle Φ between the crests and the shore-line. As the waves propagate towards the shore they become non-planar. Obtain the parametric equations

$$x = \frac{\lambda^2 g^2}{\alpha^2 \omega^4} \tanh^2 \lambda ,$$

$$z - z_0 = \frac{g^2}{\alpha^2 \omega^4} \int_0^{\lambda} \frac{(1 - \tanh^2 p \sin^2 \Phi)^{1/2}}{\tanh p \sin \Phi} \frac{d}{dp}(p^2 \tanh^2 p) \, dp ,$$

for the wave crest which passes through the shore-line at $z = z_0$. Show that near the shore-line the equation of the wave crest can be written explicitly as

$$(z - z_0)^4 \approx \left(\frac{4}{3 \sin \Phi} \right)^4 \frac{g^2}{\alpha^2 \omega^4} x^3 .$$

3.* *Wave breaking.* Ocean surface waves propagate obliquely from $x = -\infty$ on water of depth $h(x, z) = -\beta x$ towards a straight beach at $x = 0$ where they break and are dissipated. For a slowly varying depth, $\beta \ll 1$, you may assume that the dispersion relation is

$$\Omega^2 = g\kappa \tanh \kappa h \ ,$$

where $\kappa^2 = k_1^2 + k_3^2$ for the surface wavenumber (k_1, k_3). As in question 2 the frequency ω, and the component k_3 of the wavenumber along the beach, remain constant. Deduce that the shorewards component of the wavenumber, k_1, increases, and $k_1 \sim \omega(-g\beta x)^{-1/2}$ as $x \to 0$.

Find how the amplitude a of the waves varies, where $2a$ is the difference in height between the crests and the troughs of the waves. Show that if the waves break when $a\kappa = 0.1$ in a region where $\kappa h < 1$, then the point x_b at which they break is given by

$$a^2(\infty)\frac{k_1(\infty)}{\kappa(\infty)}\frac{\omega}{(-g\beta^3 x_b^3)^{1/2}} = 0.02 \ .$$

[*Hint*: Write down the solution for the free-surface, calculate the mean potential energy, and use equipartition of energy.]

4. *Sound rays in a slowly varying medium.* Deduce that for a time-independent, slowly varying, medium the frequency ω is constant at a 'ray point' moving with the group velocity. If moreover the properties of the medium are independent of two Cartesian coordinates, say x and y, deduce Snell's law that

$$\sin \alpha \propto c \ ,$$

where α is the angle between the wavenumber $\mathbf{k}$ and the z-axis, and c is the local phase speed for waves of wavenumber $\mathbf{k}$. For what type of dispersion relation is the direction of the ray parallel to $\mathbf{k}$?

Consider a dispersion relation of the form $\omega = A|\mathbf{k}|z$, where A is a constant, and let ds be an element of arc length along a ray. Show that in this case $d\alpha/ds$ is constant along a ray, and hence that each ray is the arc of a circle. Show that a wave packet moving towards the plane $z = 0$ takes an infinite time to reach it.

5. *The wave pattern generated by a duck swimming on a pseudo-fluid.* For a slowly varying, two-dimensional wave pattern of the form $A(\mathbf{x},t)\exp(\mathrm{i}\varepsilon^{-1}\theta(\mathbf{x},t))$, and a local dispersion relation $\omega = \Omega(\mathbf{k},\mathbf{x},t)$, derive the ray-tracing equations

$$\dot{k}_i = -\frac{\partial\Omega}{\partial x_i}\,, \qquad \dot{x}_i = \frac{\partial\Omega}{\partial k_i}\,, \qquad \varepsilon^{-1}\dot{\theta} = -\Omega + k_i\frac{\partial\Omega}{\partial k_i}\,, \qquad (i=1,2)\,.$$

For a homogeneous, time-independent (but not necessarily isotropic) medium, show that all rays are straight lines. When the waves have zero frequency, deduce that if the point $\mathbf{x}$ lies on a ray emanating from the origin in the direction given by a unit vector $\hat{\mathbf{c}}_g$, then

$$\theta(\mathbf{x}) = \theta(0) + \varepsilon\,\hat{\mathbf{c}}_g.\mathbf{k}|\mathbf{x}|\,.$$

Consider a duck swimming steadily with velocity V in a homogeneous pseudo-fluid. In the duck's frame, the dispersion relation is found to be

$$\Omega(k_1,k_2) = \alpha(k_1^2+k_2^2)^{\frac{1}{3}} - Vk_1\,,$$

The duck generates a steady wave pattern. By writing $(k_1,k_2) = \kappa(\cos\phi,\sin\phi)$, show that the waves satisfy

$$\kappa = \frac{\alpha^3}{V^3\cos^3\phi}\,,$$

and that the group velocity of these waves can be expressed as

$$\mathbf{c}_g = \tfrac{1}{3}V(-\cos^2\phi - 3\sin^2\phi\,, 2\sin\phi\cos\phi)\,.$$

Deduce that the waves occupy a wedge of semi-angle $\frac{1}{6}\pi$ about the negative x_1-axis. Find equation[s] describing the wave crests, and sketch the wave-crest pattern.

6. *'Reflection' and 'absorption' of internal gravity waves.* Two-dimensional internal gravity waves on a 'slowly varying' shear flow in the atmosphere satisfy the dispersion relation

$$\omega = \gamma yk + \frac{Nk}{(k^2+\ell^2)^{1/2}}\,,$$

where γ and N are positive constants, and k and ℓ are the x-and y-components of the wave number respectively. Show that as a wave packet moves, ω and k remain constant, while

$$\ell(t) = \ell_0 - \gamma kt\,,$$

where ℓ_0 is a constant. If ℓ_0 is positive, describe the motion in the [vertical] y-direction of a wave packet generated at the origin. Sketch the ray in the neighbourhood of

(*a*) $$y = -\frac{N}{\gamma}\left(\frac{1}{k} - \frac{1}{(k^2+\ell_0^2)^{1/2}}\right)\,,$$

and

(*b*) $$y = \frac{N}{\gamma(k^2+\ell_0^2)^{1/2}}\,.$$