Small particles in a viscous fluid

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Course in three parts

1. A quick course in micro-hydrodynamics
2. Sedimentation of particles
3. Rheology of suspensions

Good textbook for parts 1 & 2:
A Physical Introduction to Suspension Dynamics
by Elisabeth Guazzelli, Jeffrey F. Morris and Sylvie Pic
(Cambridge Texts in Applied Mathematics 2012).

A quick course in micro-hydrodynamics

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Continuum description: mass density $\rho(\mathbf{x}, t)$, velocity $\mathbf{u}(\mathbf{x}, t)$, etc.

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So Cauchy momentum equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \nabla\cdot\boldsymbol{\sigma} + \mathbf{F}$$

Navier-Stokes equation

Newtonian viscous fluids

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij},$$

with pressure $p(\mathbf{x}, t)$ determined globally by incompressibility
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Hence the Navier-Stokes equation (momentum for a Newtonian viscous fluid) assuming μ constant.

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Boundary conditions

$$\mathbf{u}(\mathbf{x}) \quad \text{given, or} \quad \boldsymbol{\sigma}\cdot\mathbf{n}(\mathbf{x}) \quad \text{given}$$

Small Reynolds number

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- large ν, e.g. $10^8\,\mathrm{m^2/s}$ molten glass
- e.g. $1\,\mu\mathrm{m}$ water droplet falls under gravity in air at $0.1\,\mathrm{mm/s}$, so $Re = 10^{-5}$.

Stokes equations

If $Re \ll 1$, then Stokes flow (also called "creeping flow")

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Note: Stokes theory for $Re \ll 1$ usually works for $Re < 2$.

A quick course in micro-hydrodynamics

Stokes equations

Simple properties

- Linear and instantaneous
- Reversible in time
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- Reversible in time
- Reversible in time
- Reversible in time
- Reversible in time
- Reversible in space

Flow past a sphere

More simple properties

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Flow $\mathbf{u}(\mathbf{x}, t)$ linear & instantaneous in $\mathbf{U}(t)$, also $\boldsymbol{\sigma}(\mathbf{x}, t)$, hence drag force

$$\mathbf{F}(t) = \mathbf{A}\cdot\mathbf{U}(t)$$

with $\mathbf{A}$ depending on size, shape, orientation and viscosity.

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Apply force $\mathbf{F}(t)$ in $0 \leq t \leq t_1$.
Now reverse force and its history, i.e. $\mathbf{F}(t) = -\mathbf{F}(2t_1 - t)$ in $t_1 \leq t \leq 2t_1$

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Hence cannot swim at $Re \ll 1$ by reversible flapping.

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E.g. 4. Two rigid spheres in a shear flow (possibly unequal, possibly next to a rigid wall) resume their original undisturbed streamlines after a collision.

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Uniform flow $\mathbf{U}$ past a rigid sphere of radius a.

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Uniform flow **U** past a rigid sphere of radius *a*.

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$$\mathbf{u} = \mathbf{U}\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})\left(-\frac{3a}{4r^3} + \frac{3a^3}{4r^5}\right),$$

$$p = -\frac{3a\mu\mathbf{U}\cdot\mathbf{x}}{2r^3} \quad \text{and} \quad \boldsymbol{\sigma}\cdot\mathbf{n}|_{r=a} = \frac{3\mu}{2a}\mathbf{U}.$$

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Hence the Stokes drag on the sphere is

$$\int_{r=a} \boldsymbol{\sigma}\cdot\mathbf{n}\, dS = 4\pi a^2\, \frac{3\mu}{2a}\mathbf{U} = 6\pi\mu a\mathbf{U}.$$

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The velocity and pressure fields must therefore take the forms

$$\begin{aligned}\mathbf{u}(\mathbf{x}) &= \mathbf{U}f(r) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})g(r),\\ p(\mathbf{x}) &= \mu(\mathbf{U}\cdot\mathbf{x})h(r),\end{aligned}$$

where $r = |\mathbf{x}|$, and f, g and h are functions of scalar r to be determined.

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Now

$$\frac{\partial u_i}{\partial x_j} = U_i x_j f'/r + \delta_{ij} U_n x_n g + x_i U_j g + x_i x_j U_n x_n g'/r.$$

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$$r^2 g''' + 11 r g'' + 24 g' = 0.$$

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Substituting, one finds $\alpha = 0$, -3 and -5, with associated $f = -(\alpha + 4) r^{\alpha+2}/(\alpha + 2)$ and $h = -(\alpha + 5)(\alpha + 2) r^{\alpha}$.

Hence the general solution of the assumed form linear in **u** is

$$\mathbf{u}(\mathbf{x}) = \mathbf{U}\left(-2Ar^2 + B + Cr^{-1} - \tfrac{1}{3}Dr^{-3}\right) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})\left(A + Cr^{-3} + Dr^{-5}\right),$$
$$p(\mathbf{x}) = \mu(\mathbf{U}\cdot\mathbf{x})\left(-10A + 2Cr^{-3}\right).$$

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We shall need the stress exerted across a spherical surface with unit normal $\mathbf{n} = \mathbf{x}/r$

$$\boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{U}\left(-3Ar + 2Dr^{-4}\right) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})\left(9Ar^{-1} - 6Cr^{-4} - 6Dr^{-6}\right)$$

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Applying the boundary conditions on the rigid sphere and for the far field, we find the coefficients

$$A = 0, \qquad B = 1, \qquad C = -\tfrac{3}{4}a \quad \text{and} \quad D = \tfrac{3}{4}a^3$$

For solution given earlier

Flow past a sphere

2. Solution method 2

Use a Stokes streamfunction for the axisymmetric flow

$$u_r = \frac{1}{r^2 \sin\theta} \frac{\partial \Psi}{\partial \theta} \quad \text{and} \quad u_\theta = -\frac{1}{r \sin\theta} \frac{\partial \Psi}{\partial r}.$$

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The vorticity equation (curl of the momentum equation, to eliminate the pressure) is then at low Reynolds numbers

$$\mathcal{D}^2 \mathcal{D}^2 \Psi = 0 \quad \text{where} \quad \mathcal{D}^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \right).$$

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The uniform flow at infinity has $\Psi = \frac{1}{2} U r^2 \sin^2\theta$, so one tries $\Psi = F(r) \sin^2\theta$, and finds $F = Ar^4 + Br^2 + Cr + D/r$.

3. Solution method 3

One can show (Papkovich-Neuber) that the general solution of the Stokes equation can be expressed in terms of a vector harmonic function $\boldsymbol{\phi}(\mathbf{x})$ (i.e. $\nabla^2\boldsymbol{\phi} = 0$)

$$\mathbf{u} = 2\boldsymbol{\phi} - \nabla(\mathbf{x}\cdot\boldsymbol{\phi}) \quad p = -2\mu\nabla\cdot\boldsymbol{\phi}.$$

$$\sigma_{ij} = 2\mu\left(\delta_{ij}\frac{\partial\phi_n}{\partial x_n} - x_k\frac{\partial^2\phi_k}{\partial x_i\partial x_j}\right).$$

3. Solution method 3

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$$\mathbf{u} = 2\boldsymbol{\phi} - \nabla(\mathbf{x}\cdot\boldsymbol{\phi}) \quad p = -2\mu\nabla\cdot\boldsymbol{\phi}.$$

$$\sigma_{ij} = 2\mu\left(\delta_{ij}\frac{\partial\phi_n}{\partial x_n} - x_k\frac{\partial^2\phi_k}{\partial x_i\partial x_j}\right).$$

Linearity and spherical symmetry then give

$$\boldsymbol{\phi} = A\mathbf{U}\frac{1}{r} + B\mathbf{U}\cdot\nabla\nabla\frac{1}{r},$$

with coefficients A and B to be determined by applying the boundary conditions.

Flow past a sphere

4. Solution method 4

The pressure and vorticity are harmonic functions.

Using linearity and spherical symmetry, they must take the form

$$p = \mu A \mathbf{U} \cdot \mathbf{x}/r^3 \quad \text{and} \quad \nabla \wedge \mathbf{u} = B \mathbf{U} \wedge \mathbf{x}/r^3.$$

The final step to $\mathbf{u}$ is tedious.

Sedimentation of a rigid sphere

Force balance, with densities ρ_s of sphere and ρ_f of fluid

$$\begin{array}{ccccccc} 0 & = & \rho_s \frac{4\pi}{3} a^3 g & - & \rho_f \frac{4\pi}{3} a^3 g & - & 6\pi\mu a \mathbf{U} \\ \text{no inertia} & & \text{weight} & & \text{buoyancy} & & \text{Stokes drag} \end{array}$$

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So Stokes settling velocity

$$\mathbf{U} = \frac{2\Delta\rho a^2 g}{9\mu}$$

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E.g. $1\,\mu\mathrm{m}$ sphere, $\Delta\rho = 10^3\,\mathrm{kg\,m^{-3}}$, water $\mu = 10^{-3}\,\mathrm{Pa\,s}$ gives $U = 2\mu\mathrm{m/s}$, i.e. falls through diameter in a second.

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Drag on a fluid sphere (Student exercise!)

$$\mathbf{F} = -2\pi \frac{2\mu_f + 3\mu_s}{\mu_f + \mu_s} \mu_f a U$$

Rotation of a rigid sphere

Sphere rotating at angular velocity $\boldsymbol{\Omega}$.
Flow

$$\mathbf{u}(\mathbf{x}) = \boldsymbol{\Omega} \wedge \mathbf{x} \frac{a^3}{r^3}$$

(A potential flow, so satisfies Stokes equations.)

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Flow

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(A potential flow, so satisfies Stokes equations.)

Hence couple on sphere – student exercise!

$$\mathbf{G} = 8\pi\mu a^3\boldsymbol{\Omega}$$

Stokes flow past an ellipsoid

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For a disk $a_1 \ll a_2 = a_3$

$$F_1 \sim 16\pi\mu a_2 U_1, \quad F_2 \sim \tfrac{32}{3}\mu a_2 U_2, \quad G_i \sim \tfrac{8}{3}\mu a_2 \Omega_i$$

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For a rod $a_1 \gg a_2 = a_3$, where $\ln = \ln \frac{2a_1}{a_2}$

$$F_1 \sim \frac{4\pi\mu a_1 U_1}{\ln - \frac{1}{2}}, \quad F_2 \sim \frac{8\pi\mu a_1 U_2}{\ln + \frac{1}{2}}, \quad G_1 \sim \tfrac{16}{3}\pi\mu a_1 a_2 \Omega_1, \quad G_2 \sim \frac{\frac{8}{3}\pi\mu a_1^3 \Omega_2}{\ln - \frac{1}{2}}$$

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Important conclusion Drag $\approx 6\pi\mu$ with largest diameter.

Let $\mathbf{u}^S(\mathbf{x})$ be a Stokes flow with $\mathbf{F} = 0$ in V, and let $\mathbf{u}(\mathbf{x})$ be any other incompressible flow, then

$$\int_V 2\mu e^S_{ij} e_{ij} \, dV = \int_S \sigma^S_{ij} n_j u_i \, dA.$$

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Because

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$$\sigma_{ij}^S e_{ij} = \sigma_{ij}^S \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\sigma^S{ij} u_i\right) - \underbrace{\frac{\partial \sigma_{ij}^S}{\partial x_j}}_{=\mathbf{F}=0} u_i \tag{4}$$

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Hence result by divergence theorem.

More simple properties

2. Minimum dissipation

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$$\int_V 2\mu e_{ij} e_{ij}\, dV = \int_V 2\mu e^S_{ij} e^S_{ij}\, dV$$

$$+ \int_V 2\mu (e_{ij} - e^S_{ij})(e_{ij} - e^S_{ij})\, dV + \int_V 4\mu e^S_{ij} (e_{ij} - e^S_{ij})\, dV.$$

$\longleftarrow$ positive $\longrightarrow$

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The last integral is of the form of the useful result

$$\int_V 4\mu e_{ij}^S (e_{ij} - e_{ij}^S)\, dV = \int_S 2\sigma_{ij}^S n_j (u_i - u_i^S)\, dA = 0 \quad \text{by bc}$$

Hence

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Warning: Same geometry. Cannot select geometry by minimum dissipation.

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If $\mathbf{u}^1(\mathbf{x})$ and $\mathbf{u}^2(\mathbf{x})$ are two Stokes flows in V satisfying the same boundary conditions, then minimum dissipation gives

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Hence Stokes flows are unique.

Rigid cube, sides of length $2L$, moving at $\mathbf{U}$, drag $\mathbf{F}$.

More simple properties

4. Geometric bounding

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Define second flow

$$\mathbf{u}(\mathbf{x}) = \begin{cases} \text{the Stokes flow for sphere} & \text{outside sphere,} \\ \mathbf{U} & \text{in gap.} \end{cases}$$

More simple properties

4. Geometric bounding

For this second flow

$$\int_V 2\mu e_{ij} e_{ij}\, dV = \int_{r>a} 2\mu e_{ij} e_{ij}\, dV \quad \text{because } e = 0 \text{ in gap,}$$

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$$-\mathbf{F}\cdot\mathbf{U} \le 6\pi\mu\sqrt{3}L\mathbf{U}\cdot\mathbf{U}$$

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Student exercises: bound for tetrahedron (not so tight).

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Then by the useful result

$$\int_V \mathbf{u}_1 \cdot \mathbf{f}_2 \, dV + \int_S \mathbf{U}_1 \cdot \boldsymbol{\sigma}_2 \cdot \mathbf{n} \, dA = \int_V 2\mu \mathbf{e_1} : \mathbf{e}_2 \, dV$$
$$= \int_V \mathbf{u}_2 \cdot \mathbf{f}_1 \, dV + \int_S \mathbf{U}_2 \cdot \boldsymbol{\sigma}_1 \cdot \mathbf{n} \, dA$$

More simple properties

5. Reciprocal theorem

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Then by the useful result

$$\int_V \mathbf{u}_1 \cdot \mathbf{f}_2 \, dV + \int_S \mathbf{U}_1 \cdot \boldsymbol{\sigma}_2 \cdot \mathbf{n} \, dA = \int_V 2\mu \mathbf{e_1} : \mathbf{e}_2 \, dV$$

$$= \int_V \mathbf{u}_2 \cdot \mathbf{f}_1 \, dV + \int_S \mathbf{U}_2 \cdot \boldsymbol{\sigma}_1 \cdot \mathbf{n} \, dA$$

i.e. work done by one velocity on the forces of the other is vice versa.

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Let $\mathbf{u}_2$ be Stokes flow in V with $\mathbf{f_2}$ and $\mathbf{U}_2$.

Then by the useful result

$$\int_V \mathbf{u}_1 \cdot \mathbf{f}_2 \, dV + \int_S \mathbf{U}_1 \cdot \boldsymbol{\sigma}_2 \cdot \mathbf{n} \, dA = \int_V 2\mu \mathbf{e}_1 : \mathbf{e}_2 \, dV$$
$$= \int_V \mathbf{u}_2 \cdot \mathbf{f}_1 \, dV + \int_S \mathbf{U}_2 \cdot \boldsymbol{\sigma}_1 \cdot \mathbf{n} \, dA$$

i.e. work done by one velocity on the forces of the other is vice versa.

Greens theorem in any other subject

General rigid body motion in fluid at rest a infinity,
translating at $\mathbf{U}(t)$ and rotating (about a selected point) $\boldsymbol{\Omega}(t)$.

General rigid body motion in fluid at rest a infinity, translating at $\mathbf{U}(t)$ and rotating (about a selected point) $\boldsymbol{\Omega}(t)$. By linearity and instantaneity, the force $\mathbf{F}(t)$ and couple $\mathbf{G}(t)$ (about the same selected point)

$$\begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \boldsymbol{\Omega} \end{pmatrix}$$

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The Reciprocal theorem gives for the two rigid body motions

$$\mathbf{U}_1 \cdot \mathbf{F}_2 + \mathbf{\Omega}_1 \cdot \mathbf{G}_2 = \mathbf{U}_2 \cdot \mathbf{F}_1 + \mathbf{\Omega}_2 \cdot \mathbf{G}_1$$

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True all $\mathbf{U}_1$ etc, so

$$\mathbf{A} = \mathbf{A}^T, \quad \mathbf{B} = \mathbf{C}^T \quad \text{and} \quad \mathbf{D} = \mathbf{D}^T.$$

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force due to rotation = couple due to translating.

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More simple properties

6. Reciprocal theorem – application to resistance matrix

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Need "corkscrew" feature for $\mathbf{B} \neq 0$.

More simple properties

7. Reciprocal theorem – application to Faxen's formula

A force-free sphere placed in arbitrary flow $\mathbf{u}^{\infty}(\mathbf{x})$ moves with what velocity $\mathbf{V}$? Forces which generate $\mathbf{u}^{\infty}$ kept constant.

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Applying Reciprocal theorem

$$\int \mathbf{u}_1 \cdot \underbrace{\mathbf{f}_2}_{=0} \, dV + \int \mathbf{u}_1 \cdot \underbrace{\boldsymbol{\sigma}_2 \cdot \mathbf{n}}_{=-3\mu\mathbf{u}_2/a} \, dA = \int \mathbf{u}_2 \cdot \underbrace{\mathbf{f}_1}_{=0} \, dV + \int \underbrace{\mathbf{u}_2}_{=\mathbf{U}_2} \cdot \boldsymbol{\sigma}_1 \cdot \mathbf{n} \, dA.$$

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Now look at RHS and then LHS, using $\mathbf{u}_1 = \mathbf{u}^{+} - \mathbf{u}^{\infty}$.

$$\mathrm{RHS} = \mathbf{U}_2 \cdot \left(\int \boldsymbol{\sigma}^{+} \cdot \mathbf{n} \, dA - \int \boldsymbol{\sigma}^{\infty} \cdot \mathbf{n} \, dA = 0 - 0 \right) = 0,$$

as both integrals are force on sphere.

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Hence

$$\mathrm{LHS} = -\frac{3\mu}{a} \mathbf{U}_2 \cdot \left(\int \underbrace{\mathbf{u}^+}_{=\mathbf{V}} dA - \int \mathbf{u}^\infty \, dA \right) = \mathrm{RHS} = 0$$

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For all $\mathbf{U}_2$, so velocity of sphere inserted into $\mathbf{u}^\infty(\mathbf{x})$ is

$$\mathbf{V} = \frac{1}{4\pi a^2} \int_{r=a} \mathbf{u}^\infty(\mathbf{x})\, dA$$

More simple properties

7. Reciprocal theorem – application to Faxen's formula

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$$\mathbf{u}^\infty(\mathbf{x}) = \mathbf{u}^\infty(0) + \mathbf{x}\cdot\nabla\mathbf{u}^\infty|_0 + \tfrac{1}{2}\mathbf{x}\mathbf{x} : \nabla\nabla\mathbf{u}^\infty|_0 + \cdots$$

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with higher even terms vanishing by ∇^{2n}(Stokes equations) $= 0$.

A quick course in micro-hydrodynamics

Greens function for Stokes equations
or 'Stokeslet'

For a point momentum source

$$\nabla \cdot \mathbf{u} = 0$$
$$0 = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}\delta(\mathbf{x})$$

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Solution – more important than derivation

$$\mathbf{u}(\mathbf{x}) = \mathbf{F}\cdot\mathbf{G}(\mathbf{x}) = \frac{1}{8\pi\mu}\left(\mathbf{F}\frac{1}{r} + (\mathbf{F}\cdot\mathbf{x})\mathbf{x}\frac{1}{r^3}\right)$$

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{F}\cdot\mathbf{K}(\mathbf{x}) = -\frac{3}{4\pi}\mathbf{F}\cdot\mathbf{xxx}\frac{1}{r^5}$$

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G is called the ‘Oseen tensor’.

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Already seen this in flow past a sphere:

Far from the sphere $r \gg a$, the flow is

Greens function for Stokes equations

Far field for a sphere

Far from the sphere $r \gg a$, the flow is

$$\mathbf{u} = \mathbf{U}\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})\left(-\frac{3a}{4r^3} + \frac{3a^3}{4r^5}\right),$$

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But the drag is $\mathbf{F} = -6\pi\mu a\mathbf{U}$, i.e.

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Hence far-field due to force is universal, independent of particle shape.

Greens function for Stokes equations

Integral representation

To solve

$$\nabla\cdot\mathbf{u} = 0$$
$$0 = -\nabla p + \mu\nabla^2\mathbf{u} + \mathbf{f}(\mathbf{x})$$

with boundary conditions on $\mathbf{u}(\mathbf{x})$ or $\boldsymbol{\sigma}(\mathbf{x})\cdot\mathbf{n}$.

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$$\mathbf{u}(\mathbf{x}') = \underbrace{\int_V \mathbf{G}(\mathbf{x}-\mathbf{x}')\cdot\mathbf{f}\,dV}_{\text{forces in } V} + \int_S \Big(\underbrace{\mathbf{G}(\mathbf{x}-\mathbf{x}')\cdot\boldsymbol{\sigma}(\mathbf{x})\cdot\mathbf{n}}_{\text{forces on } S} - \underbrace{(\mathbf{K}(\mathbf{x}-\mathbf{x}')\cdot\mathbf{n})\cdot\mathbf{u}}_{\text{dipoles on } S} \Big)\, dA$$

Letting $\mathbf{x}'$ in V tend onto the surface S yields an integral equation for the unknown $\mathbf{u}$ (or $\boldsymbol{\sigma}\cdot\mathbf{n}$) on S in terms of the known $\boldsymbol{\sigma}\cdot\mathbf{n}$ (or $\mathbf{u}$) on S.

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Disadvantages: Special attention needed in numerical evaluation of singular integrals, and there are often eigensolutions, e.g. constant pressure induces no flow.

Greens function for Stokes equations

Integral representation - for a suspended drop

Extension to a drop of viscosity $\lambda\mu$ surrounded by a fluid of viscosity μ.

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One knows the jump across the interface of the normal viscous stress

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$$\tfrac{1}{2}(1+\lambda)\mathbf{u}(\mathbf{x}') = \mathbf{u}^{\infty}(\mathbf{x}') \\ - \int_S \mathbf{G}(\mathbf{x}-\mathbf{x}')\cdot\gamma\kappa\mathbf{n}\,dA - (1-\lambda)\int_S \mathbf{K}(\mathbf{x}-\mathbf{x}')\cdot\mathbf{n}\cdot\mathbf{u}\,dA$$

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Slender-body theory

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or asymptotically for

$$\mathbf{f}(s_0) \sim \frac{2\pi\mu}{\ln \frac{L}{R}} \left(2\mathbf{I} - \mathbf{X}'\mathbf{X}'\right)\cdot(\mathbf{U}(s_0) - \mathbf{u}^{\infty}(\mathbf{X}(s_0)))$$

Effects of small inertia on flow past a sphere

Whitehead paradox

Flow $\mathbf{U}$ past a sphere of radius a.

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A correction $\mathbf{u}_2$ forced by this

$$-\nabla p_2 + \mu\nabla^2\mathbf{u}_2 = \rho\mathbf{u}\cdot\nabla\mathbf{u}|_{\text{Stokes}}$$

Effects of small inertia on flow past a sphere

Whitehead paradox

Flow **U** past a sphere of radius *a*.
At $Re \ll 1$, first approximation is given by Stokes flow.
Attempt to find correction as regular perturbation fails.
In far field

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would give

$$u_2 = O(\rho U^2 a/\mu)$$

which does not decay.

Look more carefully at Stokes solution. In far field

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so at $r = \mu/U$ inertial terms no longer small.

Effects of small inertia on flow past a sphere

Oseen equation

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$$\rho \mathbf{U}\cdot\nabla\mathbf{u}' = -\nabla p' + \mu\nabla^2\mathbf{u}' + \mathbf{F}\delta(\mathbf{x})$$

where in far field sphere appears a point force.

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Solve by Fourier transforms

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Solve by Fourier transforms or representation

$$\mathbf{u}' = \nabla\phi + \nu\nabla\chi - \mathbf{U}\chi \quad \text{and} \quad p' = -\rho\mathbf{U}\cdot\nabla\phi.$$

Find

$$\phi = -\frac{3a\nu}{2r}\text{(point volume source)} \quad \text{and} \quad \chi = \frac{3a}{2r}e^{\left(\frac{\mathbf{U}\cdot\mathbf{x}}{2\nu} - \frac{Ur}{2\nu}\right)}$$

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Look nearer to sphere $a \ll r \ll \nu/U$,

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Look nearer to sphere $a \ll r \ll \nu/U$, $\quad r^{-2}$ terms cancel and

Effects of small inertia on flow past a sphere

Oseen equation solved

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$$\mathbf{u}' \sim \frac{1}{r}\ \text{Stokeslet} + \mathbf{U}\frac{3Ua}{4\nu}\ \text{uniform flow}$$

Effects of small inertia on flow past a sphere

Oseen equation solved

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Hence drag increases by $1 + \frac{3}{8}Re$.

Effects of small inertia on flow past a sphere

Oseen wake

Look far from sphere $r \gg \nu/U$ near downstream axis

$$\mathbf{u}' \sim -\mathbf{U}\frac{3a}{2z}e^{-\frac{U(x^2+y^2)}{4\nu z}}$$

Effects of small inertia on flow past a sphere

Look far from sphere $r \gg \nu/U$ near downstream axis

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i.e. a wake diffusing to $r = \sqrt{\nu(t = z/U)}$ with mass flux deficit

$$\int \rho u' \, dxdy = -6\pi\mu U a \quad = \text{momentum deficit } /U$$

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Helpful idea at higher *Re*

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Impacts on time-dependent flows – Basset history wrong

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Helpful idea at higher Re

Impacts on time-dependent flows – Basset history wrong

Missing mass flux in wake goes to point source ϕ-flow.

A quick course in micro-hydrodynamics

Stokes equations

Simple properties

Flow past a sphere

More simple properties

Greens function

Effect of small inertia