Small particles in a viscous fluid

Course in three parts

1. A quick course in micro-hydrodynamics
2. Sedimentation of particles
3. Rheology of suspensions

Good textbook for parts 1 & 2:
A Physical Introduction to Suspension Dynamics
by Elisabeth Guazzelli, Jeffrey F. Morris and Sylvie Pic
(Cambridge Texts in Applied Mathematics 2012).

Part 2. Sedimentation of particles

An isolated particle

Two particles

Finite clouds

Suspensions

Fluctuations in velocity of particles

Part 2. Sedimentation of particles

An isolated particle
- A sphere
- Stokes settling velocity
- A spherical viscous drop
- Flow past an ellipsoid

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Sedimentation of an isolated sphere

Recall Stokes drag on a sphere $6\pi\mu a\mathbf{U}$

Force balance, with densities ρ_s of sphere and ρ_f of fluid

0	=	$\rho_s \frac{4\pi}{3} a^3 g$	−	$\rho_f \frac{4\pi}{3} a^3 g$	−	$6\pi\mu a\mathbf{U}$
no inertia		weight		buoyancy		Stokes drag

So Stokes settling velocity

$$\mathbf{U}_S = \frac{2\Delta\rho a^2 g}{9\mu}$$

Stokes settling velocity

$$\mathbf{U}_S = \frac{2\Delta\rho a^2 g}{9\mu}$$

E.g. $1\,\mu\mathrm{m}$ sphere, $\Delta\rho = 10^3\,\mathrm{kg\,m^{-3}}$, water $\mu = 10^{-3}\,\mathrm{Pa\,s}$ gives $U_S = 2\mu\mathrm{m\,s^{-1}}$,
i.e. falls through diameter in a second at $Re = 10^{-6}$.

Dependence $\propto a^2$, means $100\,\mu\mathrm{m}$ sphere in water would fall 10^4 times faster at $1\,\mathrm{cm\,s^{-1}}$ at $Re = 1$.

E.g. $1\,\mu\mathrm{m}$ sphere, $\Delta\rho = 10^3\,\mathrm{kg\,m^{-3}}$, air $\mu = 1.8\,10^{-5}\,\mathrm{Pa\,s}$ gives $U_S = 100\mu\mathrm{m\,s^{-1}}$ at $Re = 10^{-5}$.

Drag on a spherical drop $2\pi\mu a \frac{2\mu+3\mu_i}{\mu+\mu_i}$.

Stokes flow past an ellipsoid

For principle semi-diameters a_1, a_2, a_3 Oberbeck (1876) found

$$\text{Force}\quad F_1 = -\frac{16\pi\mu U_1}{L + a_1^2 K_2} \quad \text{and Couple} \quad G_1 = -\frac{16\pi\mu(a_2^2 + a_3^2)}{3(a_2^2 K_2 + a_3^2 K_3)}$$

where

$$L = \int_0^\infty \frac{d\lambda}{\Delta(\lambda)} \quad \text{and} \quad K_i = \int_0^\infty \frac{d\lambda}{(a_i^2 + \lambda)\Delta(\lambda)}$$

with $\Delta^2 = (a_1^2 + \lambda)(a_3^2 + \lambda)(a_3^2 + \lambda)$.

For a disk $a_1 \ll a_2 = a_3$

$$F_1 \sim 16\pi\mu a_2 U_1, \quad F_2 \sim \tfrac{32}{3}\mu a_2 U_2, \quad G_i \sim \tfrac{8}{3}\mu a_2 \Omega_i$$

For a rod $a_1 \gg a_2 = a_3$, where $\ln = \ln \frac{2a_1}{a_2}$

$$F_1 \sim \frac{4\pi\mu a_1 U_1}{\ln - \frac{1}{2}}, \quad F_2 \sim \frac{8\pi\mu a_1 U_2}{\ln + \frac{1}{2}}, \quad G_1 \sim \tfrac{16}{3}\pi\mu a_1 a_2 \Omega_1, \quad G_2 \sim \frac{\frac{8}{3}\pi\mu a_1^3 \Omega_2}{\ln - \frac{1}{2}}$$

Important conclusion Drag $\approx 6\pi\mu \times (\text{largest diameter})$.

Part 2. Sedimentation of particles

An isolated particle

Two particles
- Two equal spheres
- Two unequal spheres
- Three equal spheres
- Method of Reflections
  - Zeroth approximation
  - First reflection
  - Second reflection
  - Collecting terms

Finite clouds

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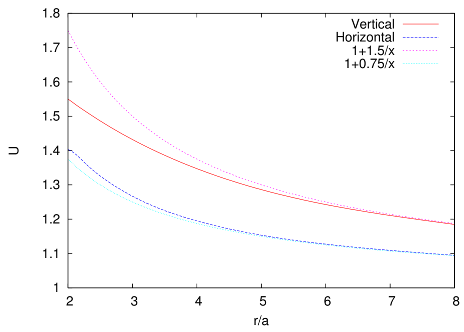
Fall of two equal spheres

Fall at constant separation by reversibility, sliding sideways.

Fall faster, up to 50% faster if close and vertical.



Approximations for large separations

$$U = U_S\left(1 + \frac{3a}{2r} + \frac{a^3}{r^3}\right), \quad U_S\left(1 + \frac{3a}{4r} + \frac{a^3}{r^3}\right)$$

Fall of two unequal spheres

Larger fall faster, so will overtake smaller.

Denser fall faster, so will overtake less dense.

Interesting interaction between two spheres with same isolated fall speeds, one smaller and denser – dance round one another.

Three equal spheres: a fast pair can catch up a slow third, interact, and a pair emerge leaving behind a different third, i.e. ever changing configuration.

Method of Reflections

An iterative method for far-field interactions between particles (use lubrication for close particles).

Zeroth approximation: Treat particles as isolated. Each create disturbances $\mathbf{u}^0(\mathbf{x})$.

First reflection: Particles react to $\mathbf{u}^0$ as if isolated. Each create disturbances $\mathbf{u}^1(\mathbf{x})$.

Second reflection: Particles react to $\mathbf{u}^1$ as if isolated. Each create disturbances $\mathbf{u}^2(\mathbf{x})$.

and so on to more reflections

Can use computer algebra to go 100 reflections.

Method of Refections: zeroth approximation

Zeroth approximation: each sphere falls as effectively isolated

$$\mathbf{u}^0 = \mathbf{u}^{01} + \mathbf{u}^{02}$$

$$\mathbf{u}^{01}(\mathbf{x}) = \mathbf{U}_1 \left(\frac{3a_1}{4r_1} + \frac{a_1^3}{4r_1^3} \right) + \frac{(\mathbf{U}_1 \cdot \mathbf{r}_1)\mathbf{r}_1}{r_1^2} \left(\frac{3a_1}{4r_1} - \frac{3a_1^3}{4r_1^3} \right)$$

with $\mathbf{U}_1$ isolated fall speed of 1, centred at $\mathbf{x}_1$, $\mathbf{r}_1 = \mathbf{x} - \mathbf{x}_1$.

And $\mathbf{u}^{02}(\mathbf{x})$ similar.

Method of Refections: first reflection

First refection: Sphere 2 sees $\mathbf{u}^{01}(\mathbf{x})$ in neighbourhood of $\mathbf{x} = \mathbf{x}_2$

$$\mathbf{u}^{01}(\mathbf{x}) = \underset{(1)}{\mathbf{u}^{01}(\mathbf{x}_2)} + \underset{(2)}{\mathbf{r}_2 \cdot \nabla \mathbf{u}^{01}|_{\mathbf{x}_2}} + \underset{(3)}{\mathbf{r}_2\mathbf{r}_2 : \nabla\nabla \mathbf{u}^{01}|_{\mathbf{x}_2}} + \dots$$

(1) dominates, a uniform flow. Sphere 2 (now force free) moves with this, i.e. additions to fall speed $O(U_1 a_1/r_{21}) + O(U_1 a_1^3/r_{21}^3)$.

(2) smaller, a linear shear flow. Sphere 2 (couple free) rotates with vorticity (antisymmetric part of ∇u^{01}). Cannot deform with straining symmetric part, so reacts with a force-dipole $O(a_2\, \mu a_2\, a_2 U_1 a_1/r_{21}^2)$.

(3) even smaller, a quadratic flow. Reacts with force quadrupole $O(a_2^2\, \mu a_2\, a_2^2 U_1 a_1/r_{21}^3)$.
Addition to fall speed by Faxen $\frac{1}{6} a_2^2 \nabla^2 u^{01} = O(a_2^2 U_1 a_1/r_{21}^3)$.

Method of Reflections: second reflection, collecting terms

Second reflection: Sphere 1 sees $\mathbf{u}^{12}(\mathbf{x})$ from dipole, quadrupole,... from sphere 2 as uniform + linear + quadratic flows
Additional fall speed due to uniform flows

$$O\left(U_1 \frac{a_1 a_2^3}{r_{21}^2} \frac{1}{r_{12}^2}\right) + O\left(U_1 \frac{a_1 a_2^5}{r_{21}^3} \frac{1}{r_{12}^3}\right)$$

Collecting contributions to fall speed

$$U_1 + U_2 \frac{a_2}{r_{12}} + U_2 \frac{a_2^3}{r_{12}^3} + U_2 \frac{a_1^2 a_2}{r_{12}^3} + U_1 \frac{a_1 a_2^2}{r_{21}^2 r_{12}^2} + U_1 \frac{a_1 a_2^5}{r_{21}^3 r_{12}^3}$$

$+1/r^8 + \ldots$ from 2nd reflection $+1/r^7 + \ldots$ from 3rd + more.

Part 2. Sedimentation of particles

A spherical cloud

A spherical cloud of particles behaves as a drop of a more viscous liquid.
Cloud radius R. Particles radius a, density difference $\Delta\rho$, volume fraction ϕ, viscosity of suspension $\mu^*(\phi)$.
Fall speed

$$\frac{2\Delta\rho\, a^2 g}{9\mu} + \frac{2\Delta\rho\, \phi R^2 g}{9\mu} \frac{2(\mu + \mu^*)}{2\mu + 3\mu^*}$$

First term from fall of individual particles is much smaller.

Due to edge being ill defined, there is a small loss of particles into the wake.

This loss produces a toroid. In a very tall tank, breaks up into two drops, process repeating (Metzner, Nicolas & Guazzelli 2007).

Boycott effect

Boycott observed in 1920 that blood settles faster if the test tubes are inclined to the vertical.

Much used in large scale industrial settlers.

Suspension only settles short vertical distance to lower inclined side and then slides slowly as a dense suspension to bottom.

Clear fluid released rises rapidly on underside of upper side.

Potential shear-flow instabilities remixing.

http://youtu.be/6I5PNdAHILI

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Suspensions
- Hindered settling
- Richardson-Zaki
- Batchelor renormalisation
  - Summing interactions in a dilute suspension
  - Divergence in naive sum
  - Batchelor's 1972 renormalisation
  - Polydispersity in sizes

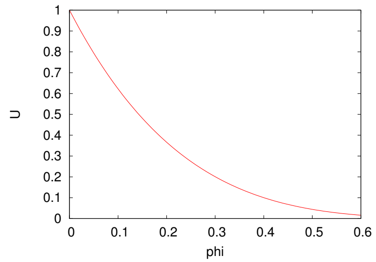
Fluctuations in velocity of particles

Settling of a uniform suspension

While two spheres settle faster than one, a suspension settles slower, often much slower.

Called 'hindered settling'. Due to 'back flow' of liquid.

Richardson-Zaki (1954) empirical correlation, for volume fraction ϕ



$$U(\phi) = U(0)(1-\phi)^{4.5}$$

Note near maximum packing reduced to $\frac{1}{20}$ – like porous media.

Summing interactions with other particles in a dilute suspension

Settling velocity of test sphere due to 2nd at distance r

$$U(r) = U_0 + \Delta U(r)$$

with Stokes velocity for isolated sphere $U_0 = 2\Delta\rho g a^2/9\mu$

Far-field form from reflections

$$\begin{aligned} \Delta U(r)/U_0 &= \tfrac{a}{r} + \tfrac{a^3}{r^3} && \text{1st reflection} \\ &\quad + \tfrac{a^4}{r^4} + \tfrac{a^6}{r^6} + \dots && \text{2nd reflection} \\ &\quad + \tfrac{a^7}{r^7} + \tfrac{a^9}{r^9} + \dots && \text{3rd reflection} \\ &\quad + \dots \end{aligned}$$

Divergence in naive sum

Naive pairwise addition of disturbances within large domain $r \le R$, with n spheres per unit volume

$$\langle \Delta U \rangle = \int_{r=2a}^{R} U_0 \left(\frac{a}{r} + \dots \right) n\, dV = O\left(U_0 \phi \frac{R^2}{a^2} \right)$$

$\phi = \frac{4\pi}{3} n a^3$ the volume fraction.

The divergence problem:

- Does mean settling velocity depend on size of domain?
- Or is it an intrinsic property independent of domain?
  i.e. is pairwise addition naive?

Batchelor's 1972 renormalisation

Batchelor's renormalisation:

$$\Delta U = \left(1 + \frac{a^2}{6}\nabla^2\right) u(x)\Big|_{\text{test sphere}} + \text{higher reflections}$$

Pairwise sum of $O(U_0 a^4/r^4)$ higher reflections is convergent

Now $\langle u \rangle_{\text{everywhere}} = 0$, so

$$\langle u \rangle_{\text{test sphere}} = -\frac{11}{2} U_0 \phi$$

$$\langle \frac{a^2}{6}\nabla^2 u \rangle_{\text{test sphere}} = \frac{1}{2} U_0 \phi$$

$$\langle \text{higher reflections} \rangle_{\text{test sphere}} = -1.55 U_0 \phi$$

Hence

$$\langle U \rangle = U_0(1 - 6.55\phi)$$

Polydispersity in sizes

Batchelor's result assumes uniform distribution of equal spheres

$$\langle U \rangle = U_0(1 - 6.55\phi)$$

Real suspension have spheres with only nearly equal radii, although same density.

Hence spheres move past one another, leading to a non-uniform distribution of pair-separations.

Hence practical result (Salin 1986)

$$\langle U \rangle = U_0(1 - 5.6\phi)$$

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