Part II continued – more details on general issues

Last time – Finite Differences

Higher orders – central, 1-sided, non-equispaced

Compact 4th order Poisson solver

Upwinding

Grids – non-Cartesian, stretched, staggered

Conservative

This time – Finite Elements

Finite Elements

Good for engineering problems with complex geometries

– 'just' need to triangulate domain

Good for elliptic, OK for parabolic, poor for hyperbolic

Good for accuracy & conservative

Poor difficult programming on unstructured grid

Poor no efficient Poisson solver on unstructured grid

Poor difficult presenting results on unstructured grid

Use packages, do not program yourself

Finite Elements = Two ideas

1. Simple representation for unknown function over the finite element
   – not point data of FD

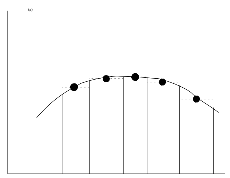
2. Weak formulation of the governing equations
   – variational statement

Representations in 1D

a. Constant elements

$$f(x) = f_i$$

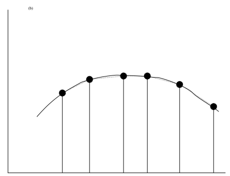
in $x_{i-1} \leq x < x_i$



b. Linear elements

$$f(x) = f_{i-1}\frac{x_i - x}{x_i - x_{i-1}} + f_i\frac{x - x_{i-1}}{x_i - x_{i-1}}$$

in $x_{i-1} \leq x < x_i$



More representations in 1D

First map element to unit interval

$$x(\xi) = x_{i-1} + (x_i - x_{i-1})\xi \quad \text{for } 0 \le \xi \le 1$$

c. Quadratic elements

$$f(x) = f_{i-1}(1-\xi)(1-2\xi) + f_{i-\frac{1}{2}}4\xi(1-\xi) + f_i\xi(2\xi-1)$$

NB: f' discontinuous at boundaries

d. Cubic elements

Obvious generalisation, but better:

$$\begin{aligned} f(x) &= f_{i-1}(1-\xi)^2(1+2\xi) + f'_{i-1}(1-\xi)^2\xi \\ &\quad + f_i\xi^2(3-2\xi) + f'_i\xi^2(1-\xi), \end{aligned}$$

Now only f'' discontinuous at boundaries – see splines later

basis functions

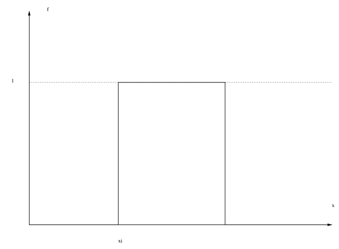
In all cases, write:

$$f(x) = \sum f_i\phi_i(x)$$

f_i amplitudes $\phi_i(x)$ basis functions, nonzero only in a few elements

For the constant elements, the basis functions are

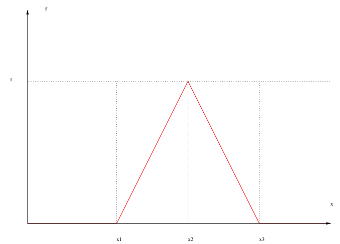
$$\phi_i(x) = \begin{cases} 1 & \text{in} \quad x_{i-1} \le x < x_i \\ 0 & \text{otherwise.} \end{cases}$$



Basis functions for linear elements

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & \text{in} \quad x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & \text{in} \quad x_i \le x \le x_{i+1} \\ 0 & \text{otherwise,} \end{cases}$$

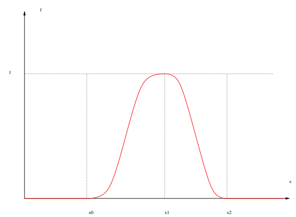
with obvious modifications for the end elements.

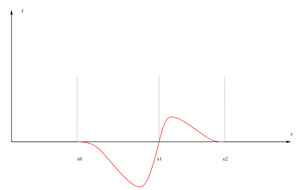


Basis functions for cubic elements

$$\phi_i(x) = \begin{cases} \frac{(x_{i+1}-x)^2(x_{i+1}+2x-3x_i)}{(x_{i+1}-x_i)^3} & \text{in} \quad x_i \le x < x_{i+1} \\ \frac{(x-x_{i-1})^2(3x_i-2x-x_{i-1})}{(x_i-x_{i-1})^3} & \text{in} \quad x_{i-1} \le x < x_i \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{\phi}_i(x) = \begin{cases} \frac{(x-x_i)(x_{i+1}-x)^2}{(x_{i+1}-x_i)^2} & \text{in} \quad x_i \le x < x_{i+1} \\ \frac{(x-x_i)(x-x_{i-1})^2}{(x_i-x_{i-1})^2} & \text{in} \quad x_{i-1} \le x < x_i \\ 0 & \text{otherwise.} \end{cases}$$





Representations in 2D

Mostly triangles, sometimes rectangles

a. Constant elements

$$f(x) = f_i \quad \text{in each triangle } i.$$

b. Linear elements Need $\ell_{12}(\mathbf{x})$ vanishing on two vertices, unity on third

$$\ell_{12}(x,y) = \frac{(x-x_1)(y_2-y_1)-(x_2-x_1)(y-y_1)}{(x_3-x_1)(y_2-y_1)-(x_2-x_1)(y_3-y_1)}.$$

Then

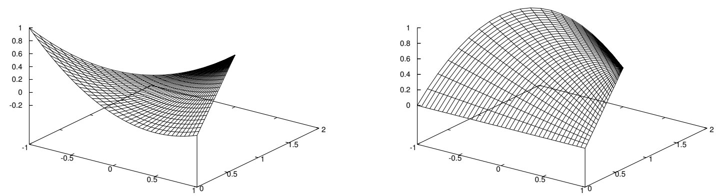
$$f(\mathbf{x}) = f_1\ell_{23}(\mathbf{x}) + f_2\ell_{31}(\mathbf{x}) + f_3\ell_{12}(\mathbf{x}).$$

Representation continuous over domain

more representations in 2D

c. Quadratic elements Values at vertices and mid-points

$$\begin{aligned} f(\mathbf{x}) \quad = \quad & f_1\ell_{23}(\mathbf{x})(2\ell_{23}(\mathbf{x})-1) \\ & +f_2\ell_{31}(\mathbf{x})(2\ell_{31}(\mathbf{x})-1) \\ & +f_3\ell_{12}(\mathbf{x})(2\ell_{12}(\mathbf{x})-1) \\ & +f_{23}4\ell_{12}(\mathbf{x})\ell_{31}(\mathbf{x}) + f_{31}4\ell_{23}(\mathbf{x})\ell_{12}(\mathbf{x}) + f_{12}4\ell_{31}(\mathbf{x})\ell_{23}(\mathbf{x}). \end{aligned}$$



more representations in 2D

d. Cubic elements

Cubic in 2D has 10 degrees of freedom:
1 constant + 2 linear + 3 quadratic + 4 cubic.

Can fit f and ∇f at vertices, plus value in centre = the 'bubble'.

e. Basis functions
In all cases, write:

$$f(x) = \sum f_i \phi_i(x)$$

For linear elements, ϕ_i is non-zero at only one vertex, vanishing on opposite sides of triangles, to form a several-sided pyramid.

Local nature → sparse coupling matrices for PDEs

more representations in 2D

f. Rectangles
Obvious constant elements

Bilinear, taking values at vertices

$$f(\mathbf{x}) = f_1\xi\eta + f_2(1-\xi)\eta + f_3\xi(1-\eta) + f_4(1-\xi)(1-\eta).$$

Continuous over domain.

Biquadratic – sum of 9 terms, each product of quadratic in separate coordinates, taking values at vertices and midpoints.

Continuous and continuous tangential derivative at boundaries.

Variational statement of Poisson problem

$$\nabla^2 f = \rho \quad \text{in volume } V$$

with boundary condition, say $\; f = g \;$ on surface S,

with $\rho(\mathbf{x})$ and $g(\mathbf{x})$ given.

Rayleigh-Ritz variational formulation: out of all those functions $f(\mathbf{x})$ that satisfy BCs, the one that minimises

$$I(f) = \int_V \left(\frac{1}{2} |\nabla f|^2 + \rho f \right) \, dV$$

also satisfies the Poisson problem.

Substitute FE representation

$$f(\mathbf{x}) = \sum f_i \phi_i(\mathbf{x})$$

Then

$$I(f) = \frac{1}{2} \sum_{ij} f_i f_j \underbrace{\int \nabla\phi_i \cdot \nabla\phi_j}_{\text{global stiffness } K_{ij}} + \sum_i f_i \underbrace{\int \rho\phi_i}_{\text{forcing } r_i}$$

Minimise over f_i

$$K_{ij} f_j + r_i = 0.$$

With these f_j, the f satisfies

$$-\int \nabla f \cdot \nabla\phi_i = \int \rho\phi_i \quad \text{for all } i,$$

i.e. satisfy PDE in all (finite) ϕ_i directions.
The weak formulation of the PDE (f can be non-C^2)

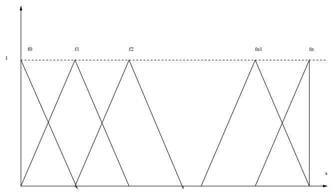
Details in 1D

$$\frac{d^2 f}{dx^2} = \rho \quad \text{in } a < x < b, \quad \text{with } f(a) = A \text{ and } f(b) = B,$$

where $\rho(x), A$ and B given.

Divide $[a, b]$ into N equal segments $h = (b - a)/N$.

Use linear finite elements with basis functions



Unknown $f(x)$ represented (BCs built in)

$$f(x) = A\phi_0(x) + B\phi_N(x) + \sum_{i=1}^{N-1} f_i \phi_i(x)$$

more details in 1D

At interior pts

$$K_{ij} = \int \nabla\phi_i \cdot \nabla\phi_j = \begin{cases} 2/h & \text{if } i = j, \\ -1/h & \text{if } i = j \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

by $\nabla\phi_i = 0, +1, -1, 0$
Take given $\rho(x)$ to be piecewise constant, then forcing

$$r_i = \int \rho(x)\phi_i = h\rho_i.$$

So equation governing unknown amplitudes f_i becomes

$$\frac{1}{h}(-f_{i-1} + 2f_i - f_{i+1}) + h\rho_i = 0 \quad \text{for } i = 1, 2, \ldots, N-1,$$

– same for the point values in the finite difference approach.

more details in 1D

Remark If evaluate r_i more accurately

$$r_i = \int \rho(x)\phi_i(x) = \rho_i + \frac{h^3}{12}\rho_i'' + O(h^5).$$

So obtain f_i to $O(h^4)$.
Yet $f(x)$ still only $O(h^2)$ in interior of elements.

Remark For non-equispaced intervals, obtain

$$\frac{1}{h_{i-\frac{1}{2}}}\left(-f_{i-1}+f_i\right) + \frac{1}{h_{i+\frac{1}{2}}}\left(f_i - f_{i+1}\right) + \frac{h_{i-\frac{1}{2}} + h_{I+\frac{1}{2}}}{2}\rho_i = 0.$$

i.e. FE approach naturally conservative.