

Last time

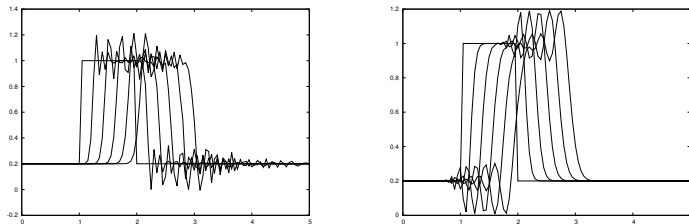
- ▶ Accuracy – first- and second-order
- ▶ Stability – CFL condition
- ▶ Longwaves – diffusion (odd), dispersion (even)

- ▶ Simplest – unstable
- ▶ Lax Friedrichs – too stable, first-order, diffusion
- ▶ Upwinding – stable, first-order, diffusion
- ▶ Crank-Nicolson, second-order, implicit, dispersion
- ▶ Lax Wendroff – second-order, explicit, dispersion
- ▶ Angled second-order, explicit, dispersion

simple advection of unsmooth ICs

High-order schemes give spurious oscillations

Angled Derivative and Lax-Wendroff



2. Simple advection of unsmooth ICs

$$u_t + cu_x = 0, \quad c > 0, \text{ const}$$

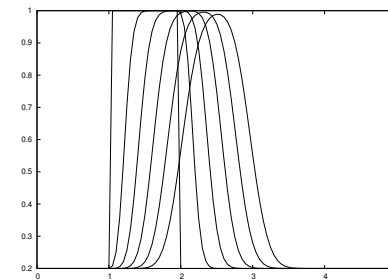
with discontinuous initial conditions

$$u = \begin{cases} 1 & 2 \leq x \leq 3 \\ 0.2 & \text{otherwise} \end{cases}$$

Problems with errors $\Delta x^2 u_{xxx}$ when $u_x = \infty$

simple advection of unsmooth ICs

Upwinding algorithm



No oscillations but lots of damping

Need new idea

3. Total Variation Diminishing

$$TV(u^n) = \sum_{\ell} |u_{\ell+1}^n - u_{\ell}^n|.$$

i.e. sum of all the differences between adjacent minima and maxima, so independent of numerical resolution.

A TVD algorithm: total variation does not increase in time

$$TV(u^{n+1}) \leq TV(u^n).$$

No spurious oscillations with new minima and maxima.

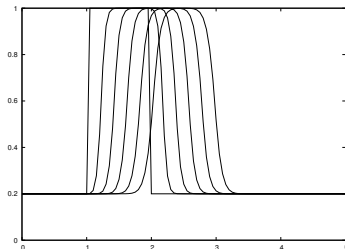
Preserves the monotonicity of a section of the solution.

eg flux-limiters – Minmod

$$a = \frac{u_{\ell+1}^n - u_{\ell}^n}{\Delta x} \text{ is to be limited by } b = \frac{u_{\ell}^n - u_{\ell-1}^n}{\Delta x}$$

$$\text{Minmod}(a, b) = \begin{cases} 0 & \text{if } ab < 0 \\ a & \text{if } ab > 0 \text{ and } |a| < |b| \\ b & \text{if } ab > 0 \text{ and } |b| < |a| \end{cases}$$

i.e. 0 in oscillation and smaller slope if monotone.



$ct = 0.0 (0.2) 1.0$
 $\Delta x = 0.05$ and $c\Delta t = 0.0125$

Flux-limiters

Idea: method = low-order (Upwind) + high-order correction (LxW)

Limiter 1: switch off correction in oscillation

Limiter 2: reduce correction if gradient changes rapidly

First reformulated in conservation form with divergence of fluxes f

$$u_{\ell}^{n+1} = u_{\ell}^n - \frac{\Delta t}{\Delta x} (f_{\ell+\frac{1}{2}}^n - f_{\ell-\frac{1}{2}}^n).$$

For Upwinding plus Lax-Wendroff correction (for $c > 0$)

$$f_{\ell+\frac{1}{2}}^n = cu_{\ell}^n + \frac{1}{2}c(\Delta x - c\Delta t)u'_{\ell+\frac{1}{2}},$$

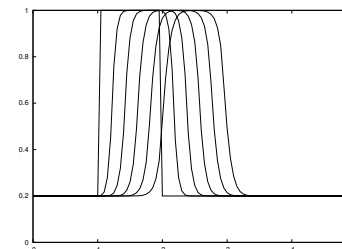
where $u'_{\ell+\frac{1}{2}} = \frac{u_{\ell+1}^n - u_{\ell}^n}{\Delta x}$ to be limited by the upstream $\frac{u_{\ell}^n - u_{\ell-1}^n}{\Delta x}$.

(If $c < 0$, the upstream side switches)

eg flux-limiters – Superbee

$$\text{Superbee}(a, b) = \begin{cases} 0 & \text{if } ab < 0, \\ a & \text{if } ab > 0 \text{ and } (|a| < \frac{1}{2}|b| \text{ or } |b| < |a| < 2|b|), \\ b & \text{if } ab > 0 \text{ and } (|b| < \frac{1}{2}|a| \text{ or } |a| < |b| < 2|a|) \end{cases}$$

i.e. 0 in oscillation and when monotone larger if less than twice smaller, otherwise smaller.



$ct = 0.0 (0.2) 1.0$
 $\Delta x = 0.05$ and $c\Delta t = 0.0125$
 slightly sharper than Minmod

4. Nonlinear advection

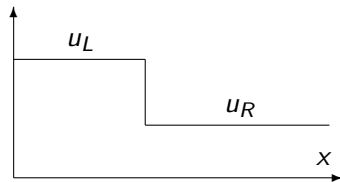
Conservative form

$$u_t + (f(u))_x = 0$$

Propagation form

$$u_t + f'(u)u_x = 0$$

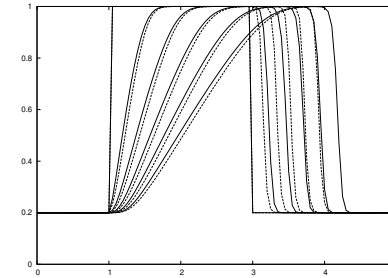
Possibility of shockwaves e.g. $f'(u) > 0$ when $u_x < 0$.



$$\text{shock speed } V = \frac{f(u_L) - f(u_R)}{u_L - u_R}$$

Conservative scheme gives correct shock speed

Case of flux $f(u) = \frac{1}{2}u^2$. Shock speed $V = 0.6$
Upwinding, $\Delta x = 0.05$, $\Delta t = 0.0125$, $ct = 0.0(0.4)2.0$



Continuous curve conservative scheme, $V = 0.59$
Dashed curve propagation scheme, $V = 0.46$

Godunov method

Three steps

- R. **Reconstruct** the solution into a simple form.
Normally a constant in each grid block, occasionally linear.
Note the discontinuities at the boundaries of the grid blocks.
- E. The simple form is **evolved** exactly.
Constant parts are advected at a constant speed.
The discontinuities are propagated as shockwaves or rarefaction waves.
The time-step must be limited by the CFL condition to stop discontinuities propagating through more than one grid block.
- A. The resulting function is **averaged** over grid blocks in preparation for step R of the next time-step.

Skip second step by using fluxes from upstream side, but which is upstream?

Godunov - upstream fluxes

For general flux $f(u)$, information propagates at $f'(u)$.
Flux $f_{\ell+\frac{1}{2}}$ from grid block ℓ to grid block $\ell + 1$

$$f_{\ell+\frac{1}{2}} = \begin{cases} f(u_\ell) & \text{if } f'(u_\ell) > 0, f'(u_{\ell+1}) > 0, \\ f(u_{\ell+1}) & \text{if } f'(u_\ell) < 0, f'(u_{\ell+1}) < 0, \\ f(u_\ell) & \text{if } f'(u_\ell) > 0, f'(u_{\ell+1}) < 0, V > 0, \\ f(u_{\ell+1}) & \text{if } f'(u_\ell) > 0, f'(u_{\ell+1}) < 0, V < 0, \\ f(u_*) & \text{if } f'(u_\ell) < 0, f'(u_{\ell+1}) > 0, \text{ where, } f'(u_*) = 0. \end{cases}$$

Last case rarefaction wave, so flux for value u_* which does not propagate

$$\text{shock speed } V = \frac{f(u_L) - f(u_R)}{u_L - u_R}$$

Further

Godonov to higher orders

Godonov is first-order in time and space, through the averaging, with large numerical diffusion $O(\Delta x^2/\Delta t)$

Replace piecewise constant by piecewise linear. Slopes can be flux-limited.

Extension to systems $\mathbf{u}(x, t)$.

– diagonalise $\mathbf{f}'(\mathbf{u})$ to find what information propagates in what direction.

Higher dimensions $u(\mathbf{x}, t)$

– Riemman solvers do not work

Finite Elements

– distribute fluxes over vertices