Computational Methods in Fluid Mechanics

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Course structure

Three parts:

- Simple Navier-Stokes problem by simple method
  – accuracy, stability, pressure
- Better treatment of general issues
  – discretisation, time-stepping, linear algebra
- Collection of special topics
  – demo FreeFem, hyperbolic, free surfaces, fast Poisson

1. The driven cavity

Incompressible Navier-Stokes

$$\nabla \cdot \mathbf{u} = 0,$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u},$$

2D, $L \times L$-box

$\mathbf{u} = 0$ on $y = 0$ and $0 < x < L$, and on $x = 0$ or L and $0 < y < L$,

and $\mathbf{u} = (U(x), 0)$ on $y = L$ and $0 < x < L$.

To find the force on the lid

$$F = \int_0^L \mu \left. \frac{\partial u}{\partial y} \right|_{y=L} dx$$

Know your physics

Before writing any code, need to think about physics
Converse, thinking about coding can deepen understanding of physics

- $\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}$
  info propagates at $\mathbf{u}$, i.e. $\delta x = u \delta t$.
- $\rho \frac{\partial \mathbf{u}}{\partial t} = \mu \nabla^2 \mathbf{u}$
  info diffuses, diffusivity $\nu = \mu/\rho$, i.e. $\delta x = \sqrt{\nu \delta t}$.
- $\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p$ with $\nabla \cdot \mathbf{u} = 0$
  info at ∞ in 0 time, i.e. speed of sound $= \infty$.
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  - $Re \ll 1$ must resolve fast diffusion of vorticity,
  - $Re \gg 1$ must resolve thin boundary layers,
  - we study $Re = 10$.

Know your PDE

What is well-posed? Equation + BCs + ICs. Wrong BC: $\nexists$ solution

- $\frac{\partial \phi}{\partial t} + u(x,t)\frac{\partial \phi}{\partial x} = f(x,t)$ – first order hyperbolic

  Well posed with

  IC $\phi(x,0)$ and

  inflow BC, e.g. at $x = a$ need $\phi(a,t)$ if $u(a,t) > 0$.

- $\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}$ – second order hyperbolic

  Well posed with

  IC $\phi(x,0)$ and $\phi_t(x,0)$ and

  BC at both ends either ϕ or ϕ_x or mixed.

- $\nabla^2 \phi = \rho$ – Laplace/Poisson equation, elliptic

  Well posed with

  BC ϕ or $\partial\phi/\partial n$ or mixed

- $\frac{\partial \phi}{\partial t} = D\frac{\partial^2 \phi}{\partial x^2}$ – Diffusion equation, parabolic

  Well posed with

  IC $\phi(x,0)$ and

  BC at both ends either ϕ or ϕ_x or mixed.

- Naming from quadratic forms

  $ax^2 + bxy + cy^2 + dx + fy + g = 0$

  $a\frac{\partial^2 \phi}{\partial x^2} + b\frac{\partial^2 \phi}{\partial x \partial y} + c\frac{\partial^2 \phi}{\partial y^2} + d\frac{\partial \phi}{\partial x} + e\frac{\partial \phi}{\partial y} + f\phi = 0$

- Numerically
  - hyperbolic – tough
  - elliptic – costly
  - parabolic – safest

Special physics – the corner

- Constant lid velocity $\mathbf{u} = (U_0, 0)$

  $\rightarrow \sigma \propto r^{-1}$ $\rightarrow F = \infty$

- Better $\mathbf{u} = (U_0 \sin \pi x/L, 0)$

  $\rightarrow \sigma \propto \ln r$ $\rightarrow F$ difficult numerically

- Therefore we take $\mathbf{u} = (U_0 \sin^2 \pi x/L, 0)$

Non-dimensionalisation

Engineers use dimensional variables in computations
but scientists do NOT.

Scale u on U_0, x and y on L, t on L/U_0 and p on ρU_0^2. Then

$$Re = \frac{\text{inertial terms } \rho U_0^2/L}{\text{viscous terms } \mu U_0/L^2} = \frac{U_0 L}{\nu}.$$

The non-dimensionalised problem

$$\nabla \cdot \mathbf{u} = 0,$$
$$\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u},$$

with BCs

$\mathbf{u} = 0$ on $y = 0$ and $0 < x < 1$, and on $x = 0$ or 1 and $0 < y < 1$

and $\mathbf{u} = (\sin^2(\pi x), 0)$ on $y = 1$ and $0 < x < 1$.

We take ICs

$\mathbf{u}(x, 0) = 0$ at $t = 0$ for $0 < x < 1$ and $0 < y < 1$.

We seek solution at $Re = 10$.

Finally the force, scaled by μU_0

$$F = \int_0^1 \left.\frac{\partial u}{\partial y}\right|_{y=1} dx.$$

Steady State vs Initial Value Problem

EJH recommends IVP, linear.

SS – nonlinear, might not exist, might be unstable.

Extrapolate slow transients to zero (Richardson).

Need not start from rest, but from SS of different Re – crude parameter continuation.

Methods for relaxing to SS $\equiv$ pseudo time-stepping.

Pressure!

Idea: time-step $\mathbf{u}(x, t)$ from t to $t + \Delta t$
using $\partial \mathbf{u}/\partial t$ from the momentum equation

But how to find ∇p?

Pressure = "Lagrangian multiplier" associated
with constraint $\nabla \cdot \mathbf{u} = 0$.

Two options:

- Find the ∇p that ensures $\nabla \cdot \mathbf{u} = 0$
  – primitive variable formulation
- Eliminate p by forming the vorticity equation
  – streamfunction-vorticity formulation

2. Streamfunction-vorticity formulation

Automatically satisfy constraint $\nabla \cdot \mathbf{u} = 0$
by using the streamfunction representation $\psi(x, y)$

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}.$$

In 2D flow vorticity is

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\nabla^2 \psi.$$

Solve as decoupled pair

1. At each t given ω, find ψ:

$$\nabla^2 \psi = -\omega$$

with $\psi = 0$ all sides.

2. With ω and now ψ known at t, find ω at $t + \Delta t$:

$$\frac{\partial \omega}{\partial t} = -\frac{\partial(\omega, \psi)}{\partial(x, y)} + \frac{1}{Re}\nabla^2 \omega$$

with ω on boundary so $\frac{\partial \psi}{\partial n}$ correct
$\rightarrow$ not quite decoupled.

Vorticity equation

Take curl of momentum equation to eliminate p

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = 0 + \frac{1}{Re}\nabla^2 \omega$$

No stretching in 2D (first term on RHS)

$$\mathbf{u} \cdot \nabla \omega = \psi_y \omega_x - \psi_x \omega_y = \frac{\partial(\omega, \psi)}{\partial(x, y)}$$

BC1: $\mathbf{u} \cdot \mathbf{n} = 0$ all sides
$\rightarrow$ sides = streamline $\quad \rightarrow \psi = 0$.

BC2: tangential velocity
$\frac{\partial \psi}{\partial y} = \sin^2 \pi x$ on top $y = 1$, $0 < x < 1$
$\frac{\partial \psi}{\partial y} = 0$ on bottom $y = 0$, $0 < x < 1$
$\frac{\partial \psi}{\partial x} = 0$ on sides $x = 0$ and 1, $0 < y < 1$