Resumé: Driven cavity, ψ–ω formulation

Poisson problem: $\nabla^2\psi = -\omega$, SOR

Vorticity evolution: $\frac{\partial\omega}{\partial t} = -\frac{\partial(\omega,\psi)}{\partial(x,y)} + \frac{1}{Re}\nabla^2\omega$

BC for ω

Timestep instability $\rightarrow \Delta t = \frac{1}{5} Re \Delta x^2$

Check $O(\Delta x^2)$ accuracy

Results, force on lid

3. Primitive variable formulation, u, v, p

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - \frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 u$$

and similar for v.

With pressure so that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Start from rest, with u and v given on the boundary.

p from where?

3.2 Pressure equation

$\nabla \cdot \mathbf{u} = 0$ all time $\rightarrow \nabla \cdot \left(\frac{\partial \mathbf{u}}{\partial t}\right) = 0$

Taking the divergence of the momentum equation

$$\underset{0}{\nabla \cdot \left(\frac{\partial \mathbf{u}}{\partial t}\right)} = -\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} \quad \underset{0}{-\mathbf{u}\cdot\nabla(\nabla\cdot\mathbf{u})} \quad -\nabla^2 p \quad \underset{0}{+\frac{1}{Re}\nabla^2(\nabla\cdot\mathbf{u})}$$

i.e.

$$\nabla^2 p = -\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

NB: Poisson problem unavoidable

But boundary condition on p?

Boundary condition

Normal component of the momentum equation at a boundary, e.g. on $x = 0$ where $u = 0$ all y and t

$$\underset{0}{\frac{\partial u}{\partial t}} + \underset{0}{u}\frac{\partial u}{\partial x} + v\underset{0}{\frac{\partial u}{\partial y}} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \underset{0}{\frac{\partial^2 u}{\partial y^2}}\right)$$

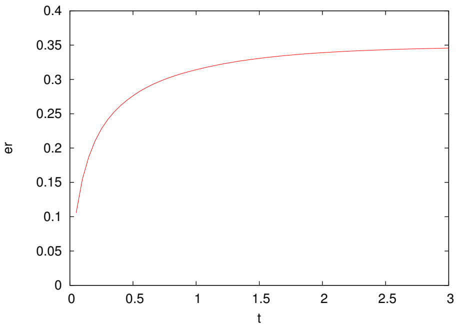
i.e.

$$\frac{\partial p}{\partial n} = \frac{1}{Re}\frac{\partial^2 u_n}{\partial n^2}$$

NB: pressure arbitrary to additive constant

Algorithm 1 (pressure equation) FAILS

Error is satisfying $\nabla \cdot \mathbf{u} = 0$ ($N = 20$) as function of t

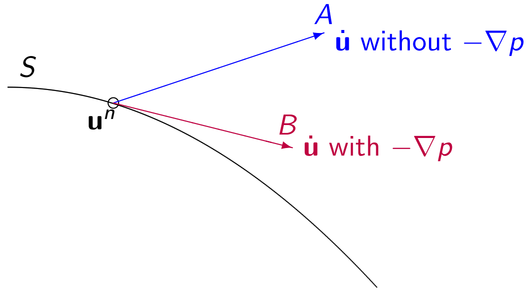


Also strange flow.
Coding error? Independent of Δt, small increase with N.

Pressure equation assumes $\nabla \cdot \mathbf{u}^n = 0$, and does not correct if untrue, so error accumulates.

3.3 Incompressibility as a constraint

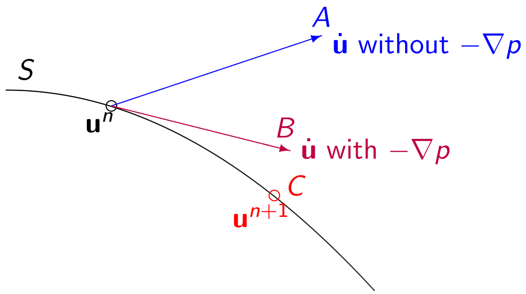
In space of all $\mathbf{u}(\mathbf{x}, t)$,
solution constrained to surface S where $\nabla \cdot \mathbf{u} = 0$



Role of ∇p is to project out component of $\partial \mathbf{u}/\partial t$ normal to surface.

. . . incompressibility as a constraint

Forward time stepping $O(\Delta t) \rightarrow$ slow drift away from surface



Avoid slow accumulation of errors by *projecting* $\mathbf{u}$ *back onto surface at C*

Implemented by split time step.

3.4 Split time step

First part (no ∇p)

$$\mathbf{u}^* = \mathbf{u}^n + \Delta t \left(-\mathbf{u}^n \cdot \boldsymbol{\nabla} \mathbf{u}^n + \frac{1}{Re} \nabla^2 \mathbf{u}^n \right).$$

Second projection part

$$\mathbf{u}^{n+1} = \mathbf{u}^* - \Delta t \boldsymbol{\nabla} p,$$

where need $\nabla \cdot \mathbf{u}^{n+1} = 0$.

So solve

$$\nabla^2 p = \frac{1}{\Delta t} \boldsymbol{\nabla} \cdot \mathbf{u}^*.$$

and then evaluate $\mathbf{u}^{n+1}$.

First part (no ∇p)

At interior points

$$
\begin{aligned}
u^*_{ij} &= u^n_{ij} \\
&+ \Delta t \left(-u^n_{ij} \frac{u^n_{i+1j} - u^n_{i-1j}}{2\Delta x} - v^n_{ij} \frac{u^n_{ij+1} - u^n_{ij-1}}{2\Delta x} \right) \\
&+ \frac{\Delta t}{Re \Delta x^2} \begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix} u^n_{ij},
\end{aligned}
$$

and a similar expression for v^*_{ij}.

Use BC on u and v.

Several algorithms for the projection step.

Projection step – algorithm 2

$$
\frac{\Delta t}{\Delta x^2} \begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix} p_{ij} = \frac{u^*_{i+1j} - u^*_{i-1j}}{2\Delta x} + \frac{v^*_{ij+1} - v^*_{ij-1}}{2\Delta x}
$$

Does not quite give the desired $\nabla \cdot \mathbf{u}^{n+1} = 0$:

has a small error which tends to zero as $\Delta x \to 0$.

3.5 Algorithm 3 - exact $\nabla \cdot \mathbf{u}^{n+1} = 0$

Now with our central differencing

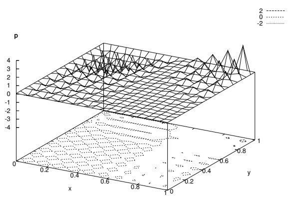
$$
\begin{aligned}
\left. \frac{\partial u^{n+1}}{\partial x} \right|_{ij} &= \frac{u^{n+1}_{i+1j} - u^{n+1}_{i-1j}}{2\Delta x} \\
&= \frac{\left(u^*_{i+1j} - \Delta t \frac{p_{i+2j} - p_{ij}}{2\Delta x} \right) - \left(u^*_{i-1j} - \Delta t \frac{p_{ij} - p_{i-2j}}{2\Delta x} \right)}{2\Delta x},
\end{aligned}
$$

and similarly for $\partial v^{n+1}/\partial y$.
Hence pressure should satisfy (recall $f'' \neq (f')'$)

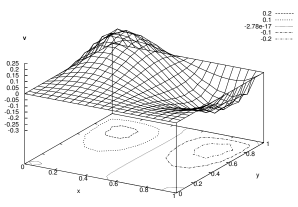
$$
\frac{\Delta t}{4\Delta x^2} \begin{pmatrix} & & 1 & & \\ & & 0 & & \\ 1 & 0 & -4 & 0 & 1 \\ & & 0 & & \\ & & 1 & & \end{pmatrix} p_{ij} = \left(\frac{u^*_{i+1j} - u^*_{i-1j}}{2\Delta x} + \frac{v^*_{ij+1} - v^*_{ij-1}}{2\Delta x} \right).
$$

Problem – spurious pressure modes

Pressure



– effect on velocity v



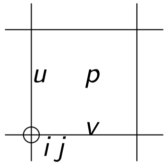
Two uncoupled solutions for pressure on odd/even $i+j$

Spurious mode $+ - + - +$ has no ∇p

Also errors 4 times larger from wide span molecule

3.6 Staggered grid – algorithm 4

New idea – a staggered grid with different variables at different locations



Write $u_{i\,j+\frac{1}{2}}$, $v_{i+\frac{1}{2}\,j}$ and $p_{i+\frac{1}{2}\,j+\frac{1}{2}}$

Good for central differencing

Boundary conditions

Boundaries coincide with mass flow BC

$$u_{0\,j+\frac{1}{2}} = u_{N\,j+\frac{1}{2}} = 0$$
$$v_{i+\frac{1}{2}\,0} = v_{i+\frac{1}{2}\,N} = 0$$

for $j = 0 \to N-1$ and $i = 0 \to N-1$ respectively

The tangential component of velocity is held half a grid block away
Use one point outside

$$v_{-\frac{1}{2}\,j} = -v_{\frac{1}{2}\,j} \quad \text{and} \quad v_{N+\frac{1}{2}\,j} = -v_{N-\frac{1}{2}\,j}$$
$$u_{i\,-\frac{1}{2}} = -u_{i\,\frac{1}{2}} \quad \text{and} \quad u_{i\,N+\frac{1}{2}} = 2\sin^2(i*\Delta x) - u_{i\,N-\frac{1}{2}}$$

for $j = 1 \to N-1$ and $i = 1 \to N-1$ respectively

Momentum equation at $i\,j+\frac{1}{2}$

First part of split time step (without pressure)

$$\begin{aligned} u^*_{i\,j+\frac{1}{2}} &= u^n_{i\,j+\frac{1}{2}} \\ &- \Delta t\, u^n_{i\,j+\frac{1}{2}} \frac{u^n_{i+1\,j+\frac{1}{2}} - u^n_{i-1\,j+\frac{1}{2}}}{2\Delta x} \\ &- \Delta t \frac{1}{4}\left(v^n_{i+\frac{1}{2}\,j} + v^n_{i-\frac{1}{2}\,j} + v^n_{i+\frac{1}{2}\,j+1} + v^n_{i-\frac{1}{2}\,j+1}\right) \frac{u^n_{i\,j+\frac{3}{2}} - u^n_{i\,j-\frac{1}{2}}}{2\Delta x} \\ &+ \frac{\Delta t}{Re\Delta x^2} \begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix} u^n_{i\,j+\frac{1}{2}}, \end{aligned}$$

Incompressibility at $i+\frac{1}{2}\,j+\frac{1}{2}$

Compact

$$\frac{\Delta t}{\Delta x^2} \begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix} p_{i+\frac{1}{2}\,j+\frac{1}{2}} = \frac{u^*_{i+1\,j+\frac{1}{2}} - u^*_{i\,j+\frac{1}{2}}}{\Delta x} + \frac{v^*_{i+\frac{1}{2}\,j+1} - v^*_{i+\frac{1}{2}\,j}}{\Delta x}.$$

Pressure boundary condition at $O(\Delta x^2)$

$$p_{-\frac{1}{2}\,j+\frac{1}{2}} = p_{\frac{1}{2}\,j+\frac{1}{2}} + \frac{1}{Re}\left(\frac{-u_{3\,j+\frac{1}{2}} + 4u_{2\,j+\frac{1}{2}} - 5u_{1\,j+\frac{1}{2}} + 2u_{0\,j+\frac{1}{2}}}{\Delta x}\right),$$

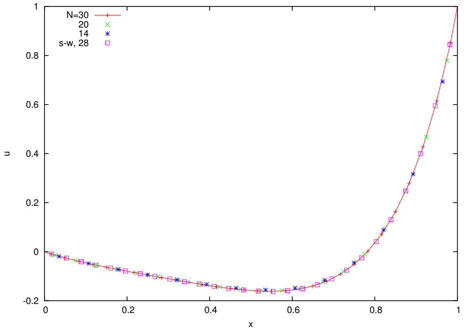
on left boundary and similar others

NB: On boundary need to advance normal component of $\mathbf{u}^n$ to nonzero $\mathbf{u}^*$ and the apply pressure projection to $\mathbf{u}^{n+1}$ back to zero, to avoid erroneously making $\partial p/\partial n = 0$

3.7 Results for algorithm 4

First check consistency of accuracy.

Steady horizontal velocity at $x = \frac{1}{2}$

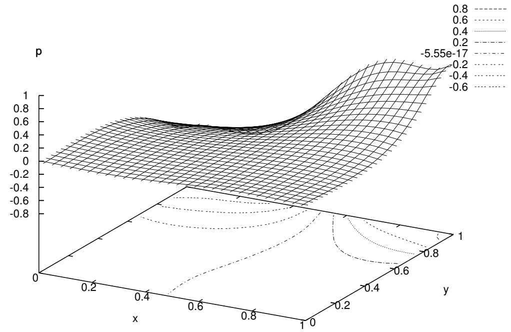


$Re = 10$ and $N = 14$, 20 and 30.
Also result from $\psi - \omega$ formulation at $N = 28$.

VERY IMPORTANT – agrees

. . . results

Pressure



$N = 30$

. . . results

Force on lid

$$F = \sum_{i=1}^{N-1} \frac{u_{i\,N+\frac{1}{2}} - u_{i\,N-\frac{1}{2}}}{\Delta x} \times \Delta x + O(\Delta x^2).$$

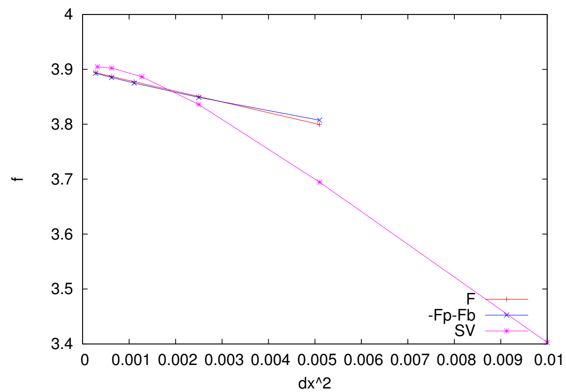
Similarly the viscous force on the bottom

Pressure force on sides from

$$\sum_{j=0}^{N-1} \frac{1}{2} \left(-p_{-\frac{1}{2}\,j+\frac{1}{2}} - p_{\frac{1}{2}\,j+\frac{1}{2}} + p_{N-\frac{1}{2}\,j+\frac{1}{2}} + p_{N+\frac{1}{2}\,j+\frac{1}{2}} \right) \times \Delta x + O(\Delta x^2).$$

. . . results

Steady force at $Re = 10$, as function of Δx^2



$$F = 3.8998 \pm 0.0002$$

and force on bottom is -0.254