

Nonlinear considerations

Nonlinear system

$$\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u})$$

Find steady states (of discretised version)

$$\mathbf{f}(\mathbf{u}) = 0$$

by **Newton** iteration (quadratic convergence)

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \delta \quad \text{with } \mathbf{f}'\delta = -\mathbf{f}(\mathbf{u}^n)$$

NB **Jacobian** \mathbf{f}' also gives linear stability info

– can test all $\text{Re}(\lambda) < 0$ without finding all λ

Eg limit cycle of Van der Pol oscillator

a nonlinear eigenvalue problem

$$\ddot{u} + \mu\dot{u}(u^2 - 1) + u = 0$$

Search for a period solution with $u(T) = u(0)$, $\dot{u}(T) = \dot{u}(0)$

Linearise (analytically) about a guess

$$u^{n+1} = u^n + \epsilon v(t), \quad T^{n+1} = T^n + \delta$$

so

$$\ddot{v} + \mu\dot{v}(u^2 - 1) + \mu\dot{u}2uv + v = 0$$

Wlog: $v(0) = 0$ and $\dot{v}(0) = 1$

Periodic if

$$u^n(T^n) + \delta\dot{u}(T^n) + \epsilon v(T^n) = u^n(0) + \epsilon(v(0) = 0)$$

$$\dot{u}^n(T^n) + \delta\ddot{u}(T^n) + \epsilon\dot{v}(T^n) = \dot{u}^n(0) + \epsilon(\dot{v}(0) = 1)$$

Solve for ϵ and δ

Find Jacobian

► Analytically – rarely, but see e.g. later
before or after discretisation

► Numerical differentiation

$$\frac{\partial f_i}{\partial u_j} \approx \frac{f_i(\mathbf{u}^n + h\mathbf{e}_j) - f_i(\mathbf{u}^n)}{h}$$

with suitably small h

► Update in last direction

$$\mathbf{f}'^{n+1} = \mathbf{f}'^n + \frac{2\mathbf{f}(\mathbf{u}^{n+1})\delta^T}{|\delta|^2}$$

may not converge

Parameter continuation

$$\mathbf{f}(\mathbf{u}, \alpha) = 0 \quad \text{with parameter } \alpha$$

Start from one solution \mathbf{u}_0 for α_0

Make a small increment to $\alpha_0 + \delta\alpha$

Find first estimate of new solution $\mathbf{u}_0 + \delta\mathbf{u}$ from

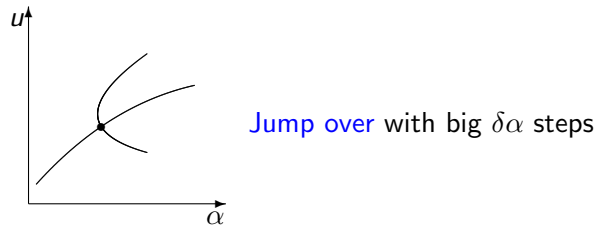
$$\delta\mathbf{u} \frac{\partial \mathbf{f}}{\partial \mathbf{u}} + \delta\alpha \frac{\partial \mathbf{f}}{\partial \alpha} = 0$$

Then Newton iterate for refined solution

But problems if $\partial \mathbf{f} / \partial \mathbf{u}$ is singular

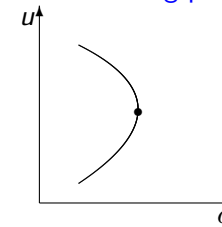
Problems when $\partial \mathbf{f} / \partial \mathbf{u}$ is singular

- ▶ Loss of **stability** of steady state of $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u}, \alpha)$
- ▶ Possible **bifurcation**.
Two eigenvalues $\text{Re}(\lambda) = 0$. Eigenvectors give directions of new solutions



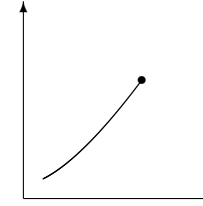
More problems when $\partial \mathbf{f} / \partial \mathbf{u}$ is singular

- ▶ Possible **turning point**



try different parameter, e.g. $|\mathbf{u}|$

- ▶ Possible **limit point**



Look at stability and nonlinear IVP

Searching for singularities of physical systems

- ▶ E.g. boundary layer equation for flow around a cylinder/sphere.
IVP blows up in a finite time
- ▶ E.g. inviscid 2D vortex sheet has finite-time singularity in the curvature of the sheet

Refine numerics

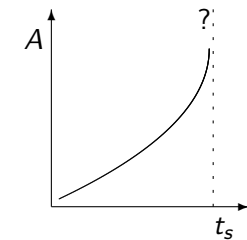
- ▶ smaller Δt
- ▶ smaller Δx
- ▶ cluster points

Only postpone singularity, never avoid

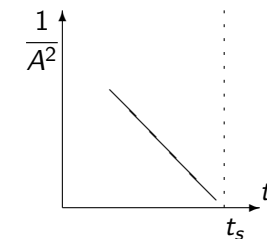
Good to have **theoretical idea** of type of singularity

– searching singularities

Do not naively plot something which blows up



If know $A \sim (T_s - t)^{-1/2}$, better plot $1/A^2$

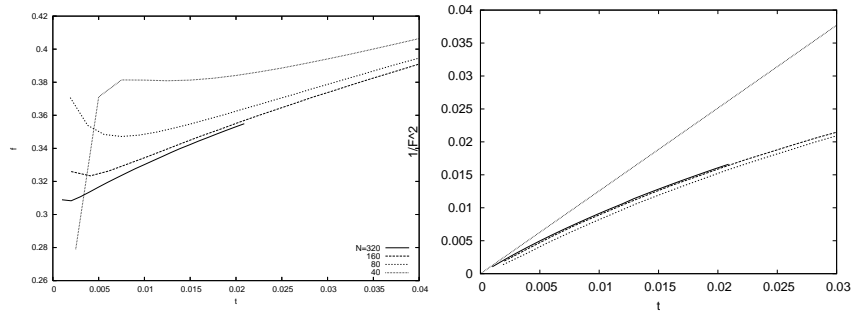


Replotting early times of driven cavity

Early times $F \sim \sqrt{Re/4\pi t}$

Original plot of $F\sqrt{t/Re}$ vs t .

Replot $1/F^2$ vs t .



Replot shows better $1/F^2 \sim 4\pi t/Re$

- searching singularities

Computer-aided algebra for power series in time

$$u(t) \sim \sum^N a_n t^n$$

Domb-Sykes plot finds t_s and α in $(t_s - t)^{-\alpha}$

$$\frac{a_n}{a_{n-1}} \sim \frac{1}{t_s} \left(1 - \frac{1-\alpha}{n} \right)$$

Conver to Padé approximant

$$\frac{\sum^K b_b t^n}{\sum^L c_n t^n}$$

and look for zeros of denominator - move with L ?