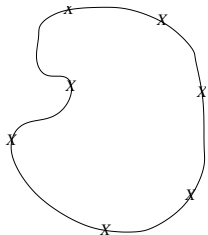
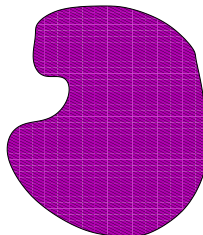


# Representation of surfaces

Marked surface



Marked volume



**Volume** good for changes in topology (drop breakup & coalescence), and cusps

**Surface** accurate curvature, used in Boundary Integral Method, can have two surfaces in one finite difference grid block

## Surfaces in 2D – a curve $\mathbf{x}(t)$

Crudest: linear segments

$$\mathbf{x}(t) = \mathbf{x}_i(i + 1 - t) + \mathbf{x}_{i+1}(t - i) \quad \text{in } i \leq t \leq i + 1.$$

Poor for curvature!

Better **cubic splines** – piecewise cubics through the data nodes  $\mathbf{x}_i$  and  $\mathbf{x}_{i+1}$ , with  $\dot{\mathbf{x}}$  and  $\ddot{\mathbf{x}}$  continuous.

$$\mathbf{x}(t) = \mathbf{x}_i(1-\tau)^2(1+2\tau) + \dot{\mathbf{x}}_i(1-\tau)^2\tau + \mathbf{x}_{i+1}\tau^2(3-2\tau) - \dot{\mathbf{x}}_{i+1}\tau^2(1-\tau),$$

where  $\tau = t - i$  in  $i \leq t \leq i + 1$ .

Requiring  $\ddot{\mathbf{x}}$  continuous at **knot**  $t = i$  gives

$$\dot{\mathbf{x}}_{i-1} + 4\dot{\mathbf{x}}_i + \dot{\mathbf{x}}_{i+1} = 3\mathbf{x}_{i+1} - 3\mathbf{x}_{i-1}.$$

a tridiagonal matrix for unknown  $\dot{\mathbf{x}}_i$ .

## surfaces in 2D

Geometry from curve  $\mathbf{x}(t)$

unit tangent  $\mathbf{t} = \dot{\mathbf{x}}/|\dot{\mathbf{x}}|$

$$\text{curvature } \kappa = \frac{\ddot{x}\dot{y} - \ddot{y}\dot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$$

Can **redistribute points**, say equi-arclength, or with penalty to weighting to concentrate on interesting feature – ‘adaptive’.

There are higher/lower order splines

Sometimes useful B-splines with compact support, e.g.  $B_3(x)$

$$\frac{1}{4} \left( (x+2)_+^3 - 4(x+1)_+^3 + 6x_+^3 - 4(x-1)_+^3 + (x-2)_+^3 \right)$$

# Surfaces in 3D

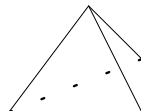
Easiest – linear approximation by triangles

Possible – bi-cubics over rectangles

Need better – radial basis functions, the generalisation of splines to higher dimensions

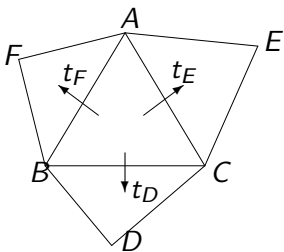
Redistribute points by adding velocity in surface for points

Diagonal swapping to increase smallest angle in a pair



Both work well with time-stepping

## surfaces in 2D – curvature over a triangle

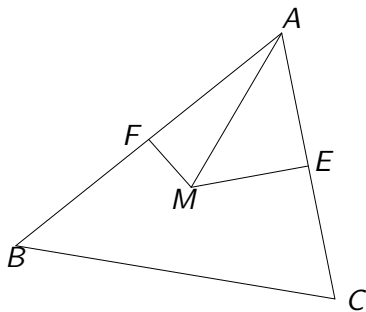
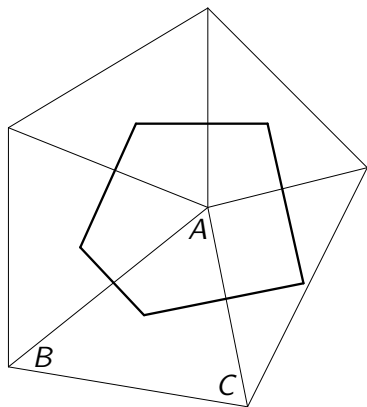


Three adjacent triangles for calculation the capillary force on triangle ABC.

$$\frac{1}{2} \left( \mathbf{n}_F \times \overrightarrow{AB} + \mathbf{n}_B \times \overrightarrow{BC} + \mathbf{n}_E \times \overrightarrow{CA} \right).$$

## surfaces in 2D – curvature for a point

Voronoi partition about marked point A using the circumcentres of the nearby triangles.



The contribution of triangle ABC is

$$\frac{1}{2} \left( \vec{AB} \cot \widehat{ACB} + \vec{AC} \cot \widehat{ABC} \right).$$

# Representation by marked volumes

## Volume of Fluid (VoF)

Phase indicator

$$c(\mathbf{x}, t) = \begin{cases} 0 & \text{in fluid 0,} \\ 1 & \text{in fluid 1} \end{cases}$$

Evolve, with good hyperbolic method (eliminate 'flotsam')

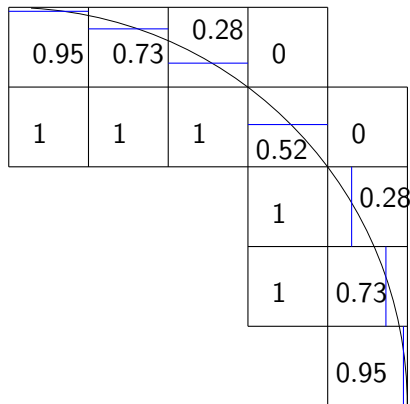
$$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c = 0.$$

Treat as a single 'effective' fluid

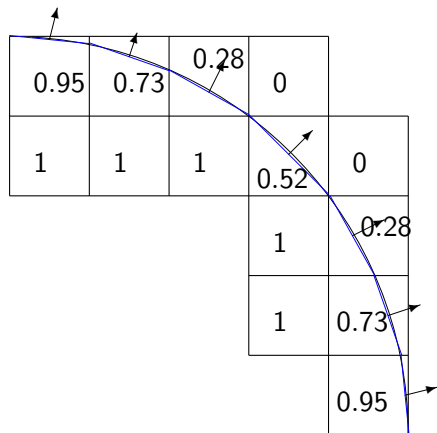
$$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c = 0.$$

Solve equations of motion in fixed Cartesian grid using conservative forms, with care in grid block containing interface

# Reconstructing the interface from $c(\mathbf{x}, t)$



(a) SLIC – Simple Linear



(b) PLIC – Piecewise Linear

For capillary force need  $\nabla c$ , but  $c$  discontinuous, so smooth/average



## reconstructing the interface from $c(\mathbf{x}, t)$

Capillary force  $\nabla \cdot \boldsymbol{\Sigma}$  in momentum conservation with

$$\boldsymbol{\Sigma} = \gamma(\mathbf{I} - \mathbf{nn})\delta(c - \frac{1}{2}) \quad \mathbf{n} = \frac{\overline{\nabla c}}{|\overline{\nabla c}|}$$

Discontinuous  $c$  is unsuitable for numerical differentiation, so smooth/average

$$\left. \frac{\partial c}{\partial x} \right|_{i+\frac{1}{2}, j+\frac{1}{2}} = \frac{1}{8\Delta x} \left( c_{i+\frac{3}{2}, j+\frac{3}{2}} + 2c_{i+\frac{3}{2}, j+\frac{1}{2}} + c_{i+\frac{3}{2}, j-\frac{1}{2}} \right. \\ \left. - c_{i-\frac{1}{2}, j+\frac{3}{2}} - 2c_{i-\frac{1}{2}, j+\frac{1}{2}} - c_{i-\frac{1}{2}, j-\frac{1}{2}} \right)$$

# Diffuse Interface method

Phase-field function  $\phi(\mathbf{x}, t)$

$$c(\mathbf{x}, t) = \begin{cases} 0 & \text{in fluid 0,} \\ 1 & \text{in fluid 1} \end{cases}$$

with a transition layer between

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = \epsilon_1 \nabla^2 \phi - \frac{1}{\epsilon_2} \frac{d}{d\phi} (\phi^2(1 - \phi)^2).$$

Layer thickness  $\sqrt{\epsilon_1 \epsilon_2}$  on time scale  $\epsilon_2$

Expensive to resolve fast thin layer which does not represent real physics

## Level Sets

Really good idea: indicator function  $\phi(\mathbf{x}, t)$  starts as roughly distance to initial surface (opposite signs on two sides), evolves

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = 0.$$

Hyperbolic, but  $\phi$  smooth, so easier

Surface found at any time from

$$\phi(\mathbf{x}, t) = 0$$

Smooth  $\phi$  gives precise position within a grid block

If contours of  $\phi$  become crowded, restart.

Possible Fast Marching Method when interface velocity is determined locally (not fluid mechanics)