Example Sheet 2

1. A force is applied to a cube at its centre in a direction normal to one flat surface. Using reversibility in space, show that the cube moves in the direction of the applied force, also without rotating. Now using linearity, deduce that in all orientations a cube of uniform density sediments vertically without rotating. [*Hints: resolve force into components, and isotropy.*]

[** What of a tetrahedron, an ellipsoid and a helix? **]

2. Show that in Stokes flow two equal spheres arbitrarily aligned fall under gravity at constant separation, i.e. neither separating nor coming closer together.

3. If the strain-rate tensor $\mathbf{e}(\mathbf{x})$ vanishes throughout a connected region, show that the flow is rigid body motion. [*Hint: first show* $\partial^2 u_1/\partial x_2 \partial x_3 \equiv 0$.]

Show that if the surface traction is specified on a bounding surface, then the Stokes flow in the interior is unique to within the addition of a rigid body motion.

4. Derive the Stokes flow outside a rotating rigid sphere

$$\mathbf{u}(\mathbf{x}) = \boldsymbol{\Omega} \times \mathbf{x} \frac{a^3}{r^3} \qquad \text{and} \quad p = 0.$$

Show that the couple exerted on the sphere is $-8\pi\mu a^3 \boldsymbol{\Omega}$.

5. If $\mathbf{A}(\mathbf{x})$ is a vector harmonic function, i.e. $\nabla^2 \mathbf{A} = 0$, show that

$$\mathbf{u} = 2\mathbf{A} - \nabla(\mathbf{A}\cdot\mathbf{x}) \qquad \text{and} \quad p = -2\mu\nabla\cdot\mathbf{A}$$

satisfy the Stokes equation. Calculate the stress tensor.

For a sphere of radius a translating at velocity $\mathbf{V}$ through a fluid which is otherwise at rest, the harmonic function takes the form

$$\mathbf{A} = \alpha a \mathbf{V} \tfrac{1}{r} + \beta a^3 \left(\mathbf{V} \cdot \nabla\right) \nabla \tfrac{1}{r},$$

(Why?) Find the constants α and β.

6. Consider a spherical bubble of radius a in a uniform flow $\mathbf{U}$. Recall the expression obtained in lectures for the Stokes flow outside a sphere of the form

$$\mathbf{u}(\mathbf{x}) = \mathbf{U} f(r) + \mathbf{x}(\mathbf{U} \cdot \mathbf{x}) g(r).$$

Applying boundary conditions on $r = a$ of no normal component of velocity and no tangential component of surface traction, find the flow $\mathbf{u}(\mathbf{x})$. Find the drag force $4\pi\mu a \mathbf{U}$.

7. Find upper and lower bounds for the couple on a tetrahedron rotating about its centre in a viscous fluid.

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8. A spherical annulus of incompressible viscous liquid occupies the region $R_1(t) < r < R_2(t)$ between two free surfaces on which pressures (normal traction) $P_1(t)$ and $P_2(t)$ are applied. The resulting flow is spherically symmetric. Show (neglecting inertia and surface tension)

$$\frac{d}{dt}\left(R_1^3\right) = \frac{\pi(P_1 - P_2)}{\mu V} R_1^3 \left(R_1^3 + 3V/4\pi\right),$$

where V is the constant volume of the liquid. [*Hints:* $u_r = A/r^2$ *(why?) and* $\sigma_{rr} = -p + 2\mu\partial u/\partial r$ *in this flow.*]

Show that if $P_1 - P_2$ is maintained positive and constant, then R_1 becomes infinite in a finite time. What happens if $P_1 - P_2$ is maintained negative and constant.

9. Fluid is contained in the region $-\alpha < \theta < \alpha$ between two rigid hinged plates. Thus the velocity components in plane polar coordinates satisfy

$$u_r = 0, \qquad u_\theta = \mp\omega r \qquad \text{on} \quad \theta = \pm\alpha.$$

Neglecting inertia forces, show that a solution to the Stokes problem may be found in the form

$$\psi = \tfrac{1}{2}\omega r^2 g(\theta)$$

and find the function $g(\theta)$. Deduce the pressure field $p(r, \theta)$. Discuss the limitations of the model. [*Does denominator of* g *vanish?*]

10. Viscous fluid is contained between two planes $y = \pm a$ and a two-dimensional flow with streamfunction $\psi(x, y)$ is generated by some agency (e.g. a rotating cylinder) near $x = y = 0$. It is required to find the form of the flow field for large positive x. Find the general solution of $\nabla^4\psi = 0$ of the form

$$\psi = f(y)e^{-kx} \qquad \mathrm{Re}\, k > 0,$$

for which $f(y)$ is an even function of y, and hence show that k is determined by the equation

$$2ka + \sin 2ka = 0.$$

Show that this equation as no real roots. The equation has complex roots, that with the smallest real part being $2ka = 4.2 \pm 2.3i$. Sketch the streamlines of the flow.