Example Sheet 3

1. A rigid sphere of radius a falls through a fluid of viscosity μ under gravity towards a horizontal rigid plane. Use lubrication theory to show that, when the minimum gap h_0 is very small, the speed of approach of the sphere is

$$h_0 W/6\pi\mu a^2,$$

where W is the weight of the sphere corrected for buoyancy.

2. Oil is forced by a pressure difference Δp through the narrow gap between two parallel circular cylinders of radius a with axes $2a + b$ apart. Show that, provided $b \ll a$ and $\rho b^3 \Delta p \ll \mu^2 a$, the volume flux is approximately

$$\frac{2b^{5/2}\Delta p}{9\pi a^{1/2}\mu}$$

when the cylinders are fixed.

Show also that when the two cylinders rotate with angular velocities Ω_1 and Ω_2 in opposite directions, the change in the volume flux is

$$\tfrac{2}{3}ab(\Omega_1 + \Omega_2).$$

3. A viscous fluid coats the outer surface of a cylinder of radius a which rotates with angular velocity Ω about its axis which is horizontal. The angle θ is measured from the horizontal on the rising side. Show that the volume flux per unit length $Q(\theta, t)$ is related to the thickness $h(\theta, t)$ of the fluid layer by

$$Q = \Omega a h - \frac{g}{3\nu} h^3 \cos\theta,$$

and deduce an evolution equation for $h(\theta, t)$.

Consider now the possibility of a steady state with $Q =$ const, $h = h(\theta)$. Show that a steady solution with $h(\theta)$ continuous and 2π-periodic exists only if

$$\Omega a > (9Q^2 g/4\nu)^{1/3}.$$

4. A two-dimensional drop $h(x, t)$ spreads on a horizontal table. Assuming that the drop has become a thin layer, find how the drops spreads. [It is not possible to integrate the volume in closed form.]

5. The walls of a channel are porous and separated by a distance d. Fluid is driven through the channel by a pressure gradient $G = -\partial p/\partial x$, and at the same time suction is applied to one wall of the channel providing a cross flow with uniform transverse component of velocity V, fluid being supplied at this rate at the other wall. Find and sketch the steady velocity and vorticity distributions in the fluid (i) when $Vd/\nu \ll 1$ and (ii) when $Vd/\nu \gg 1$.

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6. Viscous fluid fills an annulus $a < r < b$ between a long stationary cylinder $r = b$ and a long cylinder $r = a$ rotating at angular velocity Ω. Find the axisymmetric velocity field, ignoring end effects.

Suppose now that the two cylinders are porous, and a pressure difference is applied so that there is a radial flow $-Va/r$. Find the new steady flow around the cylinder when $Va/\nu < 2$ and $Va/\nu > 2$. Comment on the flow structure when $Va/\nu \gg 1$.

Find the torque that must be applied to maintain the motion.

7. Starting from the Navier-Stokes equations for incompressible viscous flow with conservative forces, obtain the vorticity equation

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}.$$

Interpret the terms in the equation.

At time $t = 0$ a concentration of vorticity is created along the z-axis, with the same circulation Γ around the axis at each z. The fluid is viscous and incompressible, and for $t > 0$ has only an azimuthal velocity v, say. Show that there is a similarity solution of the form $vr/\Gamma = f(\eta)$, where $r = (x^2 + y^2)^{1/2}$ and η is a suitable similarity variable. Further show that all conditions are satisfied by

$$f(\eta) = \tfrac{1}{2\pi}(1 - e^{-\eta}), \qquad \eta = r^2/4\nu t.$$

Show also that the total vorticity in the flow remains constant at Γ for all $t > 0$. Sketch v as a function of r.

8. Calculate the vorticity $\boldsymbol{\omega}$ associated with the velocity field

$$\mathbf{u} = (-\alpha x - yf(r,t),\ -\alpha y + xf(r,t),\ 2\alpha z)\,,$$

where α is a positive constant, and $f(r,t)$ depends on $r = (x^2 + y^2)^{1/2}$ and time t. Hence show that the velocity field represents a dynamically possible motion if $f(r,t)$ satisfies

$$2f + r\frac{\partial f}{\partial r} = A\gamma(t)e^{-\gamma(t)r^2},$$

where

$$\gamma(t) = \frac{\alpha}{2\nu}\left(1 \pm e^{-2\alpha(t-t_0)}\right)^{-1},$$

and A and t_0 are constants.

Show that in the case where the minus sign is taken γ is approximately $1/[4\nu(t-t_0)]$ when t only just exceeds t_0. Which terms in the vorticity equation dominate when this approximation holds?