

Resumé: Driven cavity, ψ - ω formulation

Poisson problem: $\nabla^2\psi = -\omega$, SOR

Vorticity evolution: $\frac{\partial\omega}{\partial t} = -\frac{\partial(\omega,\psi)}{\partial(x,y)} + \frac{1}{Re}\nabla^2\omega$

BC for ω

Timestep instability $\rightarrow \Delta t = \frac{1}{5}Re\Delta x^2$

Check $O(\Delta x^2)$ accuracy

Results, force on lid

3. Primitive variable formulation, u, v, p

$$\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} - \frac{\partial p}{\partial x} + \frac{1}{Re} \nabla^2 u$$

and similar for v .

With pressure so that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Start from rest, with u and v given on the boundary.

p from where?

Boundary condition

Normal component of the momentum equation at a boundary, e.g. on $x = 0$ where $u = 0$ all y and t

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

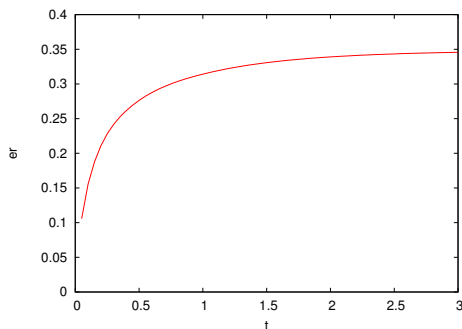
i.e.

$$\frac{\partial p}{\partial n} = \frac{1}{Re} \frac{\partial^2 u_n}{\partial n^2}$$

NB: pressure arbitrary to additive constant

Algorithm 1 (pressure equation) FAILS

Error is satisfying $\nabla \cdot \mathbf{u} = 0$ ($N = 20$) as function of t



Also strange flow.

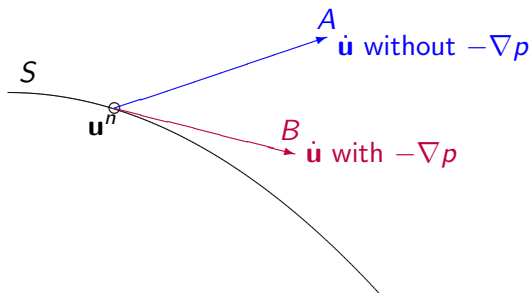
Coding error? Independent of Δt , small increase with N .

Pressure equation assumes $\nabla \cdot \mathbf{u}^n = 0$, and does not correct if untrue, so error accumulates.

3.3 Incompressibility as a constraint

In space of all $\mathbf{u}(\mathbf{x}, t)$,

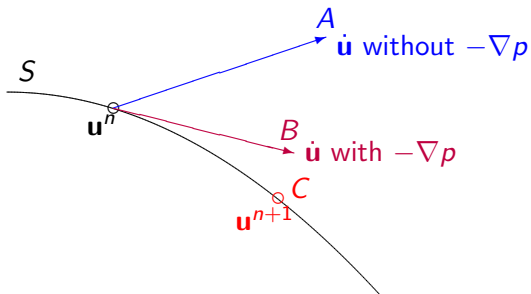
solution constrained to surface S where $\nabla \cdot \mathbf{u} = 0$



Role of ∇p is to **project out** component of $\partial \mathbf{u} / \partial t$ normal to surface.

... incompressibility as a constraint

Forward time stepping $O(\Delta t) \rightarrow$ slow drift away from surface



Avoid slow accumulation of errors by *projecting \mathbf{u} back onto surface at C*

Implemented by split time step.

3.4 Split time step

First part (no ∇p)

$$\mathbf{u}^* = \mathbf{u}^n + \Delta t \left(-\mathbf{u}^n \cdot \nabla \mathbf{u}^n + \frac{1}{Re} \nabla^2 \mathbf{u}^n \right).$$

Second projection part

$$\mathbf{u}^{n+1} = \mathbf{u}^* - \Delta t \nabla p,$$

where need $\nabla \cdot \mathbf{u}^{n+1} = 0$.

So solve

$$\nabla^2 p = \frac{1}{\Delta t} \nabla \cdot \mathbf{u}^*.$$

and then evaluate \mathbf{u}^{n+1} .

First part (no ∇p)

At interior points

$$\begin{aligned} u_{ij}^* &= u_{ij}^n \\ &+ \Delta t \left(-u_{ij}^n \frac{u_{i+1j}^n - u_{i-1j}^n}{2\Delta x} - v_{ij}^n \frac{u_{ij+1}^n - u_{ij-1}^n}{2\Delta x} \right) \\ &+ \frac{\Delta t}{Re\Delta x^2} \begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix} u_{ij}^n, \end{aligned}$$

and a similar expression for v_{ij}^* .

Use BC on u and v .

Several algorithms for the projection step.

Projection step – algorithm 2

$$\frac{\Delta t}{\Delta x^2} \begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix} p_{ij} = \frac{u_{i+1j}^* - u_{i-1j}^*}{2\Delta x} + \frac{v_{ij+1}^* - v_{ij-1}^*}{2\Delta x}$$

Does not quite give the desired $\nabla \cdot \mathbf{u}^{n+1} = 0$:

has a small error which tends to zero as $\Delta x \rightarrow 0$.

3.5 Algorithm 3 - exact $\nabla \cdot \mathbf{u}^{n+1} = 0$

Now with our central differencing

$$\begin{aligned}\left. \frac{\partial u^{n+1}}{\partial x} \right|_{ij} &= \frac{u_{i+1j}^{n+1} - u_{i-1j}^{n+1}}{2\Delta x} \\ &= \frac{\left(u_{i+1j}^* - \Delta t \frac{p_{i+2j} - p_{ij}}{2\Delta x} \right) - \left(u_{i-1j}^* - \Delta t \frac{p_{ij} - p_{i-2j}}{2\Delta x} \right)}{2\Delta x},\end{aligned}$$

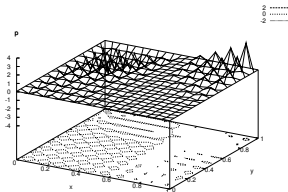
and similarly for $\partial v^{n+1}/\partial y$.

Hence pressure should satisfy (recall $f'' \neq (f')'$)

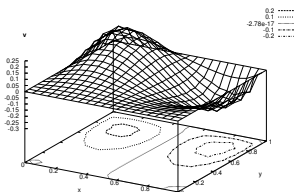
$$\frac{\Delta t}{4\Delta x^2} \begin{pmatrix} & & 1 & & \\ & & 0 & & \\ 1 & 0 & -4 & 0 & 1 \\ & & 0 & & \\ & & 1 & & \end{pmatrix} p_{ij} = \left(\frac{u_{i+1j}^* - u_{i-1j}^*}{2\Delta x} + \frac{v_{ij+1}^* - v_{ij-1}^*}{2\Delta x} \right).$$

Problem – spurious pressure modes

Pressure



– effect on velocity v



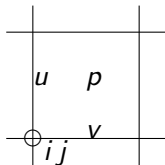
Two uncoupled solutions for pressure on odd/even $i + j$

Spurious mode $+ - + - +$ has no ∇p

Also errors 4 times larger from wide span molecule

3.6 Staggered grid – algorithm 4

New idea – a staggered grid with different variables at different locations



Write $u_{i,j+\frac{1}{2}}$, $v_{i+\frac{1}{2},j}$ and $p_{i+\frac{1}{2},j+\frac{1}{2}}$

Good for central differencing

Boundary conditions

Boundaries coincide with mass flow BC

$$u_{0j+\frac{1}{2}} = u_{Nj+\frac{1}{2}} = 0$$

$$v_{i+\frac{1}{2}0} = v_{i+\frac{1}{2}N} = 0$$

for $j = 0 \rightarrow N - 1$ and $i = 0 \rightarrow N - 1$ respectively

The tangential component of velocity is held half a grid block away
Use one point outside

$$v_{-\frac{1}{2}j} = -v_{\frac{1}{2}j} \quad \text{and} \quad v_{N+\frac{1}{2}j} = -v_{N-\frac{1}{2}j}$$

$$u_{i-\frac{1}{2}} = -u_{i\frac{1}{2}} \quad \text{and} \quad u_{iN+\frac{1}{2}} = 2 \sin^2(i * \Delta x) - u_{iN-\frac{1}{2}}$$

for $j = 1 \rightarrow N - 1$ and $i = 1 \rightarrow N - 1$ respectively

Momentum equation at $ij + \frac{1}{2}$

First part of split time step (without pressure)

$$\begin{aligned}u_{ij+\frac{1}{2}}^* &= u_{ij+\frac{1}{2}}^n \\&- \Delta t u_{ij+\frac{1}{2}}^n \frac{u_{i+1j+\frac{1}{2}}^n - u_{i-1j+\frac{1}{2}}^n}{2\Delta x} \\&- \Delta t \frac{1}{4} \left(v_{i+\frac{1}{2}j}^n + v_{i-\frac{1}{2}j}^n + v_{i+\frac{1}{2}j+1}^n + v_{i-\frac{1}{2}j+1}^n \right) \frac{u_{ij+\frac{3}{2}}^n - u_{ij-\frac{1}{2}}^n}{2\Delta x} \\&+ \frac{\Delta t}{Re\Delta x^2} \begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix} u_{ij+\frac{1}{2}}^n,\end{aligned}$$

Incompressibility at $i + \frac{1}{2}j + \frac{1}{2}$

Compact

$$\frac{\Delta t}{\Delta x^2} \begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix} p_{i+\frac{1}{2}j+\frac{1}{2}} = \frac{u_{i+1j+\frac{1}{2}}^* - u_{ij+\frac{1}{2}}^*}{2\Delta x} + \frac{v_{i+\frac{1}{2}j+1}^* - v_{i+\frac{1}{2}j}^*}{2\Delta x}.$$

Pressure boundary condition at $O(\Delta x^2)$

$$p_{-\frac{1}{2}j+\frac{1}{2}} = p_{\frac{1}{2}j+\frac{1}{2}} + \frac{1}{Re} \left(\frac{-u_{3j+\frac{1}{2}} + 4u_{2j+\frac{1}{2}} - 5u_{1j+\frac{1}{2}} + 2u_{0j+\frac{1}{2}}}{\Delta x} \right),$$

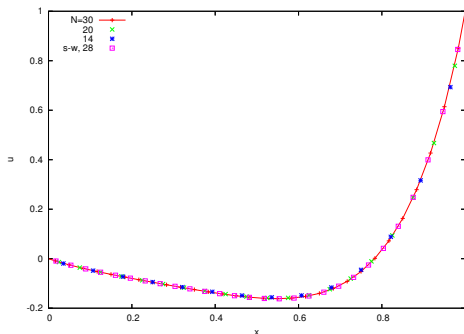
on left boundary and similar others

NB: On boundary need to advance normal component of \mathbf{u}^n to nonzero \mathbf{u}^* and then apply pressure projection to \mathbf{u}^{n+1} back to zero, to avoid erroneously making $\partial p / \partial n = 0$

3.7 Results for algorithm 4

First check consistency of accuracy.

Steady horizontal velocity at $x = \frac{1}{2}$

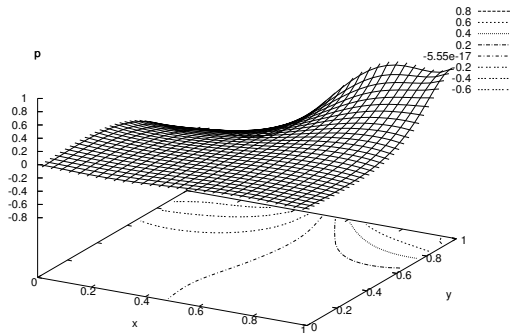


$Re = 10$ and $N = 14, 20$ and 30 .

Also result from $\psi - \omega$ formulation at $N = 28$.

VERY IMPORTANT – agrees

Pressure



$$N = 30$$

... results

Force on lid

$$F = \sum_{i=1}^{N-1} \frac{u_{iN+\frac{1}{2}} - u_{iN-\frac{1}{2}}}{\Delta x} \times \Delta x + O(\Delta x^2).$$

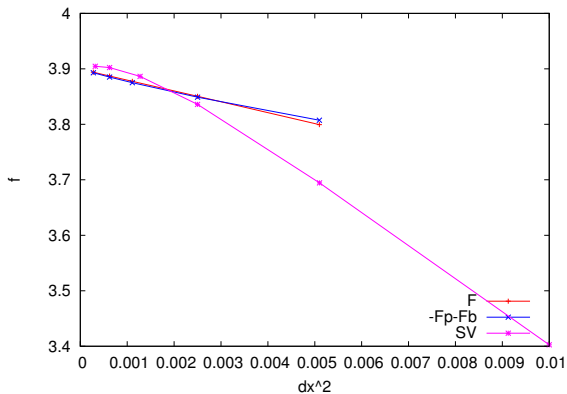
Similarly the viscous force on the bottom

Pressure force on sides from

$$\sum_{j=0}^{N-1} \frac{1}{2} \left(-p_{-\frac{1}{2}j+\frac{1}{2}} - p_{\frac{1}{2}j+\frac{1}{2}} + p_{N-\frac{1}{2}j+\frac{1}{2}} + p_{N+\frac{1}{2}j+\frac{1}{2}} \right) \times \Delta x + O(\Delta x^2).$$

... results

Steady force at $Re = 10$, as function of Δx^2



$$F = 3.8998 \pm 0.0002$$

and force on bottom is -0.254