

Hyperbolic equations

Avoid numerically

- ▶ Advection + diffusion

OK if $\Delta x < D/U$. Then $D\Delta t < \Delta x^2$ gives $U\Delta t < \Delta x$

- ▶ Advection + reaction

OK if $\Delta x < U\tau$. Then $U\Delta t < \Delta x$ gives $\Delta t < \tau$

- ▶ Pure Advection

- ▶ Problem 1 conserve past numerical errors
- ▶ Problem 2 shocks = unresolved boundary layers = rarefaction waves and discontinuities ← unfriendly to high-order schemes

Hint: Reformulate with characteristics, i.e. Lagrangian

1. Simple smooth advection

$$u_t + cu_x = 0,$$

and smooth initial condition

$$u(x) = \begin{cases} 4(x-1)^2(2-x)^2 & \text{in } 1 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Take c constant, > 0 .

Generalise to $c(x)$, $c(x, u)$ and vector $\mathbf{u}(\mathbf{x}, t)$

Finite differences easier for cooperation of spatial and temporal discretisations.

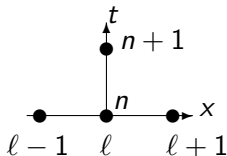
Write

$$u_\ell^n = u(x = \ell\Delta x, t = n\Delta t).$$

1.1 Simplest - unstable

First-order in time, central second-order in space

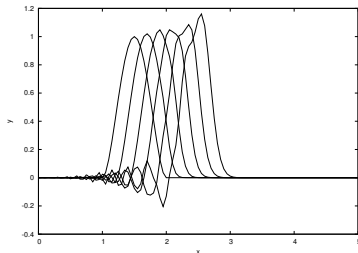
$$\frac{u_l^{n+1} - u_l^n}{\Delta t} = -c \frac{u_{l+1}^n - u_{l-1}^n}{2\Delta x}$$



$$ct = 0.0 (0.2) 1.0$$

$$\Delta x = 0.05$$

$$c\Delta t = 0.0125$$



Unstable

Stability analysis

Set $u_\ell^n = A^n e^{ik\ell\Delta x}$, (Fourier wave). To find $A(k)$

Algorithm $\rightarrow A = 1 - i\mu \sin \theta$ with $\mu = \frac{c\Delta t}{\Delta x}$ and $\theta = k\Delta x$.

Then $|A| > 1$ all μ ,

i.e. **unstable** all Δt .

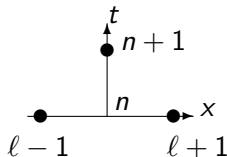
Most unstable = short wave zigzag $\theta = \frac{\pi}{2}$

with $|A| = \sqrt{1 + \mu^2}$

i.e. $u \sim (1 + \mu^2)^{t/2\Delta t}$.

1.2 Lax-Friedricks – too stable

Replacing u_ℓ^n in the time derivative
by average $\frac{1}{2}(u_{\ell+1}^n + u_{\ell-1}^n)$.



$$u_\ell^{n+1} = \frac{1}{2} \left(1 - \frac{c\Delta t}{\Delta x} \right) u_{\ell+1}^n + \frac{1}{2} \left(1 + \frac{c\Delta t}{\Delta x} \right) u_{\ell-1}^n.$$

Stability analysis $u_\ell^n = A^n e^{ik\ell\Delta x}$

$$A = \cos \theta - i\mu \sin \theta \quad \text{with } \mu = \frac{c\Delta t}{\Delta x} \text{ and } \theta = k\Delta x$$

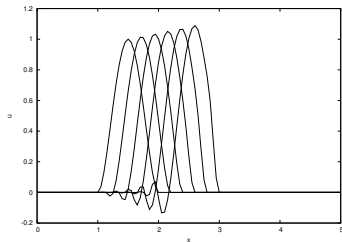
i.e. stable $|A| < 1$ all θ if

$$\mu = \frac{c\Delta t}{\Delta x} < 1 \quad \text{CFL condition (Courant-Friedricks-Lewy)}$$

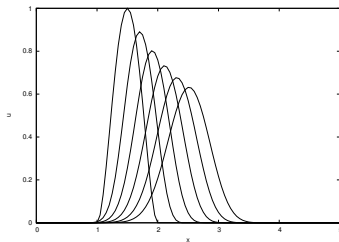
Information propagates less than Δx in Δt

Lax-Friedricks – too stable

Plots $ct = 0.0(0.2) 1.0$, $\Delta x = 0.05$



unstable $\mu = c\Delta t/\Delta x = 1.1$



stable $\mu = 0.5$

Stable but very damped

Longwave error analysis

Taylor series

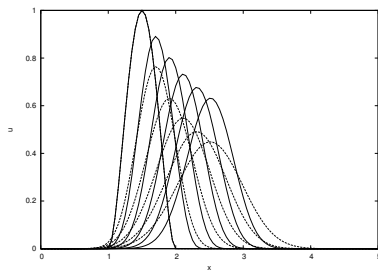
$$u_{\ell+1}^n = u_{\ell}^n + \Delta x u_x^n + \frac{1}{2} \Delta x^2 u_{xx}^n + \dots,$$

$$u_{\ell}^{n+1} = u_{\ell}^n + \Delta t u_t^n + \frac{1}{2} \Delta t^2 u_{tt}^n + \dots$$

Algorithm + Lax trick $u_{tt} = c^2 u_{xx}$

$$u_t = -cu_x + \frac{1}{2}(1 - \mu^2) \frac{\Delta x^2}{\Delta t} u_{xx}.$$

Numerical diffusion



$ct = 0.0(0.2) 1.0$

for $\Delta x = 0.05$

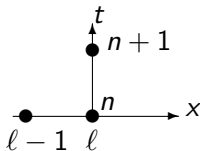
continuous $c\Delta t = 0.025$

dashed $c\Delta t = 0.0125$

NB numerical diffusion ↗ as $\Delta t \rightarrow 0$

1.3 Upwinding – avoid downstream influence

$$\frac{u_\ell^{n+1} - u_\ell^n}{\Delta t} = -c \frac{u_\ell^n - u_{\ell-1}^n}{\Delta x}$$

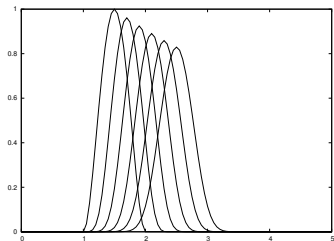


Stability

$$|A|^2 = 1 - 4\mu(1 - \mu) \sin^2 \frac{\theta}{2}, \quad \text{i.e. stable if } \mu < 1$$

Longwave error analysis

$$u_t = -cu_x + \frac{1}{2}(1 - \mu)c\Delta x u_{xx}.$$



$$ct = 0.0 \ (0.2) \ 1.0$$

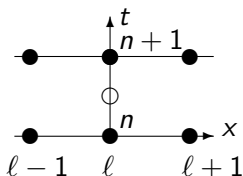
$$\Delta x = 0.05, \ \Delta t = 0.25$$

numerical diffusivity bounded

as $\Delta t \rightarrow 0$

1.4 Crank-Nicolson – second-order, implicit

Central difference about mid-point $(\ell, n + \frac{1}{2})$



$$\frac{u_{\ell}^{n+1} - u_{\ell}^n}{\Delta t} = -\frac{c\Delta t}{4\Delta x} (u_{\ell+1}^{n+1} - u_{\ell-1}^{n+1} + u_{\ell+1}^n - u_{\ell-1}^n).$$

Stability

$$A = \frac{1 - \frac{1}{2}i\mu \sin \theta}{1 + \frac{1}{2}i\mu \sin \theta}.$$

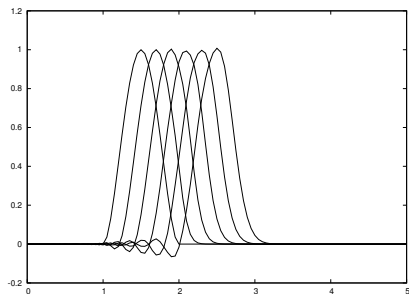
i.e. $|A| = 1$ all μ : **stable** with no damping (?accurate large μ ?)

Crank-Nicolson

Longwave error analysis

$$u_t = -cu_x - \frac{1}{12}(2 - \mu^2)c\Delta x^2 u_{xxx}.$$

u_{xxx} means numerical dispersion



$$ct = 0.0 \ (0.2) \ 1.0$$

$$\Delta x = 0.05, \ c\Delta t = 0.025$$

Slower short waves at the trailing edge

1.5 Lax-Wendroff – second-order, explicit

Upwinding corrected by subtracting off leading error

$$\frac{1}{2}(1 - \mu)c\Delta x [u_{xx} \approx (u_{\ell+1}^n - 2u_{\ell}^n + u_{\ell-1}^n)/\Delta x^2]$$

and rearranging

$$u_{\ell}^{n+1} = u_{\ell}^n - \frac{c\Delta t}{2\Delta x} (u_{\ell+1}^n - u_{\ell-1}^n) + \frac{c^2\Delta t^2}{2\Delta x^2} (u_{\ell+1}^n - 2u_{\ell}^n + u_{\ell-1}^n)$$

Stability

$$|A|^2 = 1 - 4\mu^2(1 - \mu^2) \sin^4 \frac{1}{2}\theta,$$

stable if $\mu < 1$ (CFL)

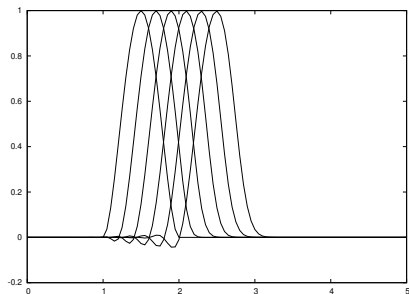
Longwave errors

$$u_t = -cu_x - \frac{1}{6}(1 - \mu^2)c\Delta x^2 u_{xxx}.$$

again numerical dispersion

Lax-Wendroff

$$u_{\ell}^{n+1} = u_{\ell}^n - \frac{c\Delta t}{2\Delta x} (u_{\ell+1}^n - u_{\ell-1}^n) + \frac{c^2\Delta t^2}{2\Delta x^2} (u_{\ell+1}^n - 2u_{\ell}^n + u_{\ell-1}^n)$$

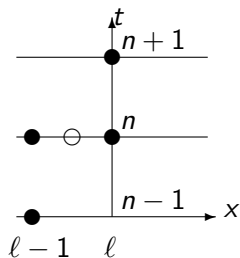


$$ct = 0.0 \ (0.2) \ 1.0$$
$$\Delta x = 0.05, \ c\Delta t = 0.025$$

Slower short waves at the trailing edge

1.6 Angled derivative – second-order, explicit, 3-level

Central difference about mid-point $(\ell - \frac{1}{2}, n)$



$$(u_t)_{\ell-\frac{1}{2}}^n = \frac{1}{2} \left(\frac{u_{\ell-1}^n - u_{\ell-1}^{n-1}}{\Delta t} + \frac{u_{\ell}^{n+1} - u_{\ell}^n}{\Delta t} \right) = -c(u_x)_{\ell-\frac{1}{2}}^n = c \frac{u_{\ell}^n - u_{\ell-1}^n}{\Delta x}.$$

Re-arranging

$$u_{\ell}^{n+1} = \left(1 - \frac{2c\Delta t}{\Delta x} \right) (u_{\ell}^n - u_{\ell-1}^n) + u_{\ell-1}^{n-1}.$$

Angled derivative

Stability

$$\left(Ae^{i\theta/2}\right)^2 - 2i(1 - 2\mu) \sin \frac{1}{2}\theta \left(Ae^{i\theta/2}\right) - 1 = 0,$$

stable $\mu < 1$, but spurious (stable) second mode

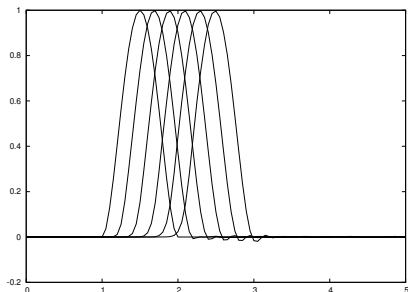
Longwave errors

$$u_t = -cu_x + \frac{1}{12}(1 - \mu)(1 - 2\mu)c\Delta x^2 u_{xxx}.$$

numerical dispersion, vanishes at $\mu = \frac{1}{2}$ (when exact!)

Angled derivative

$$\text{Start } u_\ell^1 = u_\ell^0 - \frac{c\Delta t}{2\Delta x}(u_{\ell+1}^0 - u_{\ell-1}^0)$$



$$ct = 0.0 \ (0.2) \ 1.0$$
$$\mu = 0.3$$

Conclusions for smooth problems

CFL stability: $\mu = \frac{c\Delta t}{\Delta x} < 1$ (typically)

Odd-order schemes \rightarrow numerical diffusion

i.e. spreading and decay

Even-order schemes \rightarrow numerical dispersion

i.e. spurious (typically trailing) oscillations