Hyperbolic equations

Avoid numerically

- Advection + diffusion

  OK if $\Delta x < D/U$. Then $D\Delta t < \Delta x^2$ gives $U\Delta t < \Delta x$

- Advection + reaction

  OK if $\Delta x < U\tau$. Then $U\Delta t < \Delta x$ gives $\Delta t < \tau$

- Pure Advection
  - Problem 1 conserve past numerical errors
  - Problem 2 shocks = unresolved boundary layers = rarefaction waves and discontinuities $\leftarrow$ unfriendly to high-order schemes

Hint: Reformulate with characteristics, i.e. Lagrangian

1. Simple smooth advection

$$u_t + cu_x = 0,$$

and smooth initial condition

$$u(x) = \begin{cases} 4(x-1)^2(2-x)^2 & \text{in} \quad 1 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Take c constant, > 0.

Generalise to $c(x)$, $c(x, u)$ and vector $\mathbf{u}(\mathbf{x}, t)$

Finite differences easier for cooperation of spatial and temporal discretisations.

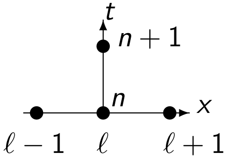
Write

$$u_\ell^n = u(x = \ell\Delta x, t = n\Delta t).$$

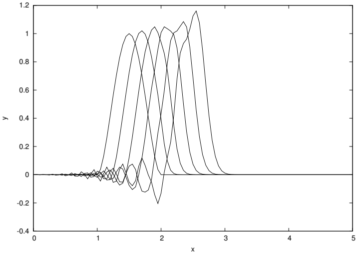
1.1 Simplest - unstable

First-order in time, central second-order in space

$$\frac{u_\ell^{n+1} - u_\ell^n}{\Delta t} = -c\,\frac{u_{\ell+1}^n - u_{\ell-1}^n}{2\Delta x}$$



$ct = 0.0\,(0.2)\,1.0$
$\Delta x = 0.05$
$c\Delta t = 0.0125$



Unstable

Stability analysis

Set $u_\ell^n = A^n e^{ik\ell\Delta x}$, (Fourier wave). To find $A(k)$

Algorithm $\to$ $A = 1 - i\mu\sin\theta$ with $\mu = \frac{c\Delta t}{\Delta x}$ and $\theta = k\Delta x$.

Then $|A| > 1$ all μ,

i.e. unstable all Δt.

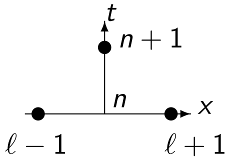
Most unstable = short wave zigzag $\theta = \frac{\pi}{2}$

with $|A| = \sqrt{1+\mu^2}$

i.e. $u \sim (1+\mu^2)^{t/2\Delta t}$.

1.2 Lax-Friedricks – too stable

Replacing u_ℓ^n in the time derivative
by average $\frac{1}{2}(u_{\ell+1}^n + u_{\ell-1}^n)$.



$$u_\ell^{n+1} = \frac{1}{2}\left(1 - \frac{c\Delta t}{\Delta x}\right) u_{\ell+1}^n + \frac{1}{2}\left(1 + \frac{c\Delta t}{\Delta x}\right) u_{\ell-1}^n.$$

Stability analysis $u_\ell^n = A^n e^{ik\ell\Delta x}$

$$A = \cos\theta - i\mu\sin\theta \quad \text{with } \mu = \frac{c\Delta t}{\Delta x} \text{ and } \theta = k\Delta x$$

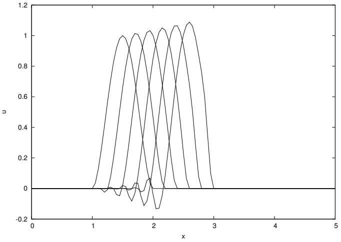
i.e. stable $|A| < 1$ all θ if

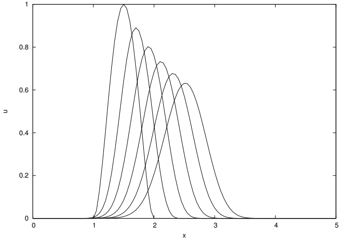
$$\mu = \frac{c\Delta t}{\Delta x} < 1 \quad \text{CFL condition (Courant-Friedricks-Lewy)}$$

Information propagates less than Δx in Δt

Lax-Friedricks – too stable

Plots $ct = 0.0\,(0.2)\,1.0,\ \Delta x = 0.05$





unstable $\mu = c\Delta t/\Delta x = 1.1$

stable $\mu = 0.5$

Stable but very damped

Longwave error analysis

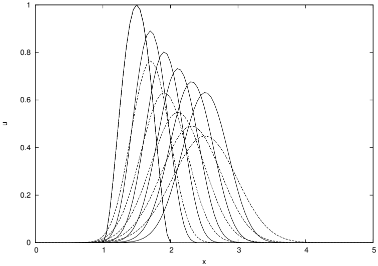
Taylor series

$$u_{\ell+1}^{n} = u_{\ell}^{n} + \Delta x \, u_{x}{}_{\ell}^{n} + \tfrac{1}{2}\Delta x^{2} \, u_{xx}{}_{\ell}^{n} + \dots,$$
$$u_{\ell}^{n+1} = u_{\ell}^{n} + \Delta t \, u_{t}{}_{\ell}^{n} + \tfrac{1}{2}\Delta t^{2} \, u_{t}{}_{\ell}^{n} + \dots.$$

Algorithm + Lax trick $u_{tt} = c^2 u_{xx}$

$$u_t = -c u_x + \tfrac{1}{2}(1-\mu^2)\frac{\Delta x^2}{\Delta t} u_{xx}.$$

Numerical diffusion

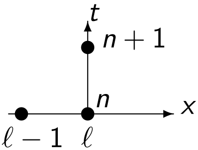


$ct = 0.0\,(0.2)\,1.0$
for $\Delta x = 0.05$
continuous $c\Delta t = 0.025$
dashed $c\Delta t = 0.0125$

NB numerical diffusion $\nearrow$ as $\Delta t \to 0$

1.3 Upwinding – avoid downstream influence

$$\frac{u_\ell^{n+1} - u_\ell^n}{\Delta t} = -c\,\frac{u_\ell^n - u_{\ell-1}^n}{\Delta x}$$

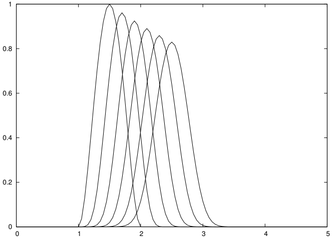


Stability

$$|A|^2 = 1 - 4\mu(1-\mu)\sin^2\tfrac{\theta}{2}, \quad \text{i.e. stable if } \mu < 1$$

Longwave error analysis

$$u_t = -c u_x + \tfrac{1}{2}(1-\mu)c\Delta x\, u_{xx}.$$



$ct = 0.0\,(0.2)\,1.0$

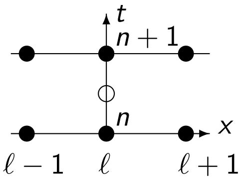
$\Delta x = 0.05,\ \Delta t = 0.25$

numerical diffusivity bounded

as $\Delta t \to 0$

1.4 Crank-Nicolson – second-order, implicit

Central difference about mid-point $(\ell, n+\frac{1}{2})$



$$\frac{u_\ell^{n+1} - u_\ell^n}{\Delta t} = -\frac{c\Delta t}{4\Delta x}\left(u_{\ell+1}^{n+1} - u_{\ell-1}^{n+1} + u_{\ell+1}^{n} - u_{\ell-1}^{n}\right).$$

Stability

$$A = \frac{1 - \frac{1}{2} i\mu \sin\theta}{1 + \frac{1}{2} i\mu \sin\theta}.$$

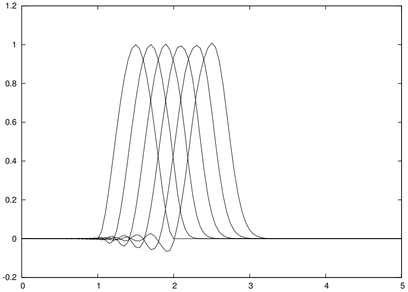
i.e. $|A| = 1$ all μ: stable with no damping (?accurate large μ?)

Crank-Nicolson

Longwave error analysis

$$u_t = -cu_x - \frac{1}{12}(2 - \mu^2)c\Delta x^2\, u_{xxx}.$$

u_{xxx} means numerical dispersion



$ct = 0.0\,(0.2)\,1.0$
$\Delta x = 0.05,\ c\Delta t = 0.025$

Slower short waves at the trailing edge

1.5 Lax-Wendroff – second-order, explicit

Upwinding corrected by subtracting off leading error

$$\tfrac{1}{2}(1-\mu)c\Delta x\ \left[u_{xx} \approx (u^n_{\ell+1} - 2u^n_\ell + u^n_{\ell-1})/\Delta x^2\right]$$

and rearranging

$$u^{n+1}_\ell = u^n_\ell - \frac{c\Delta t}{2\Delta x}\left(u^n_{\ell+1} - u^n_{\ell-1}\right) + \frac{c^2\Delta t^2}{2\Delta x^2}\left(u^n_{\ell+1} - 2u^n_\ell + u^n_{\ell-1}\right)$$

Stability

$$|A|^2 = 1 - 4\mu^2(1-\mu^2)\sin^4\tfrac{1}{2}\theta,$$

stable if $\mu < 1$ (CFL)

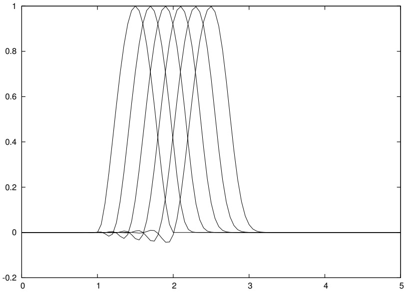
Longwave errors

$$u_t = -cu_x - \tfrac{1}{6}(1-\mu^2)c\Delta x^2 u_{xxx}.$$

again numerical dispersion

Lax-Wendroff

$$u_\ell^{n+1} = u_\ell^n - \frac{c\Delta t}{2\Delta x}\left(u_{\ell+1}^n - u_{\ell-1}^n\right) + \frac{c^2\Delta t^2}{2\Delta x^2}\left(u_{\ell+1}^n - 2u_\ell^n + u_{\ell-1}^n\right)$$

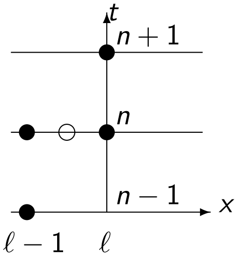


$ct = 0.0\,(0.2)\,1.0$
$\Delta x = 0.05,\ c\Delta t = 0.025$

Slower short waves at the trailing edge

1.6 Angled derivative – second-order, explicit, 3-level

Central difference about mid-point $(\ell - \frac{1}{2}, n)$



$$(u_t)^n_{\ell-\frac{1}{2}} = \frac{1}{2}\left(\frac{u^n_{\ell-1} - u^{n-1}_{\ell-1}}{\Delta t} + \frac{u^{n+1}_{\ell} - u^n_{\ell}}{\Delta t}\right) = -c(u_x)^n_{\ell-\frac{1}{2}} = c\frac{u^n_{\ell} - u^n_{\ell-1}}{\Delta x}.$$

Re-arranging

$$u^{n+1}_{\ell} = \left(1 - \frac{2c\Delta t}{\Delta x}\right)\left(u^n_{\ell} - u^n_{\ell-1}\right) + u^{n-1}_{\ell-1}.$$

Angled derivative

Stability

$$\left(Ae^{i\theta/2}\right)^2 - 2i(1-2\mu)\sin\tfrac{1}{2}\theta\left(Ae^{i\theta/2}\right) - 1 = 0,$$

stable $\mu < 1$, but spurious (stable) second mode

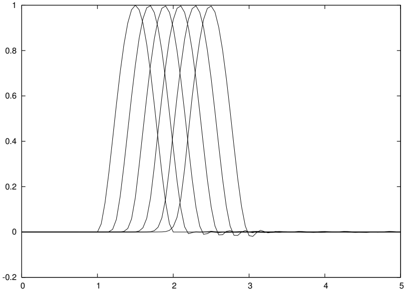
Longwave errors

$$u_t = -cu_x + \tfrac{1}{12}(1-\mu)(1-2\mu)c\Delta x^2\, u_{xxx}.$$

numerical dispersion, vanishes at $\mu = \frac{1}{2}$ (when exact!)

Angled derivative

Start $u_\ell^1 = u_\ell^0 - \frac{c\Delta t}{2\Delta x}(u_{\ell+1}^0 - u_{\ell-1}^0)$



$ct = 0.0\,(0.2)\,1.0$
$\mu = 0.3$

Conclusions for smooth problems

CFL stability: $\mu = \frac{c\Delta t}{\Delta x} < 1$ (typically)

Odd-order schemes $\rightarrow$ numerical diffusion

i.e. spreading and decay

Even-order schemes $\rightarrow$ numerical dispersion

i.e spurious (typically trailing) oscillations