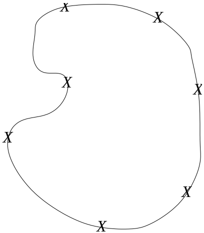
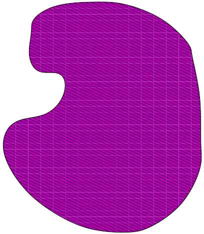
Representation of surfaces

Marked surface



Marked volume



Volume good for changes in topology (drop breakup & coalescence), and cusps

Surface accurate curvature, used in Boundary Integral Method, can have two surfaces in one finite difference grid block

Surfaces in 2D – a curve $\mathbf{x}(t)$

Crudest: linear segments

$$\mathbf{x}(t) = \mathbf{x}_i(i+1-t) + \mathbf{x}_{i+1}(t-i) \quad \text{in } i \leq t \leq i+1.$$

Poor for curvature!

Better cubic splines – piecewise cubics through the data nodes $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$, with $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$ continuous.

$$\mathbf{x}(t) = \mathbf{x}_i(1-\tau)^2(1+2\tau) + \dot{\mathbf{x}}_i(1-\tau)^2\tau + \mathbf{x}_{i+1}\tau^2(3-2\tau) - \dot{\mathbf{x}}_{i+1}\tau^2(1-\tau),$$

where $\tau = t - i$ in $i \leq t \leq i+1$.

Requiring $\ddot{\mathbf{x}}$ continuous at knot $t = i$ gives

$$\dot{\mathbf{x}}_{i-1} + 4\dot{\mathbf{x}}_i + \dot{\mathbf{x}}_{i+1} = 3\mathbf{x}_{i+1} - 3\mathbf{x}_{i-1}.$$

a tridiagonal matrix for unknown $\dot{\mathbf{x}}_i$.

surfaces in 2D

Geometry from curve $\mathbf{x}(t)$

$$\text{unit tangent } \mathbf{t} = \dot{\mathbf{x}}/|\dot{\mathbf{x}}|$$

$$\text{curvature } \kappa = \frac{\ddot{x}\dot{y} - \ddot{y}\dot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$$

Can redistribute points, say equi-arclength, or with penalty to weighting to concentrate on interesting feature – 'adpative'.

There are higher/lower order splines
Somtimes useful B-splines with compact support, e.g. $B_3(x)$

$$\tfrac{1}{4}\left((x+2)_+^3 - 4(x+1)_+^3 + 6x_+^3 - 4(x-1)_+^3 + (x-2)_+^3\right)$$

Surfaces in 3D

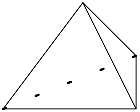
Easiest – linear approximation by triangles

Possible – bi-cubics over rectangles

Need better – radial basis functions, the generalisation of splines to higher dimensions

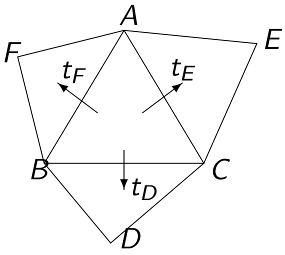
Redistribute points by adding velocity in surface for points

Diagonal swapping to increase smallest angle in a pair



Both work well with time-stepping

surfaces in 2D – curvature over a triangle

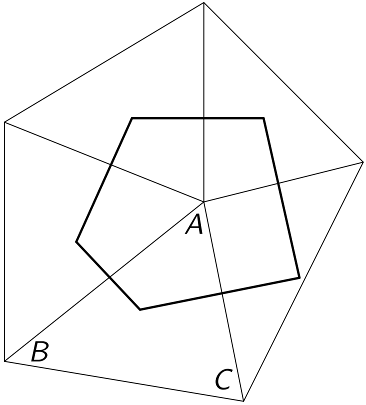


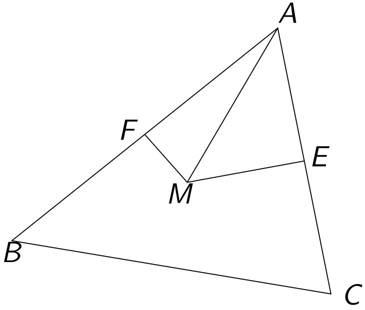
Three adjacent triangles for calculation the capillary force on triangle ABC.

$$\frac{1}{2}\left(\mathbf{n}_F \times \overrightarrow{AB} + \mathbf{n}_B \times \overrightarrow{BC} + \mathbf{n}_E \times \overrightarrow{CA}\right).$$

surfaces in 2D – curvature for a point

Voronoi partition about marked point A using the circumcentres of the nearby triangles.





The contribution of triangle ABC is

$$\frac{1}{2}\left(\overrightarrow{AB}\cot\widehat{ACB}+\overrightarrow{AC}\cot\widehat{ABC}\right).$$

Representation by marked volumes

Volume of Fluid (VoF)

Phase indicator

$$c(\mathbf{x}, t) = \begin{cases} 0 & \text{in fluid 0,} \\ 1 & \text{in fluid 1} \end{cases}$$

Evolve, with good hyperbolic method (elimiate 'flotsam')

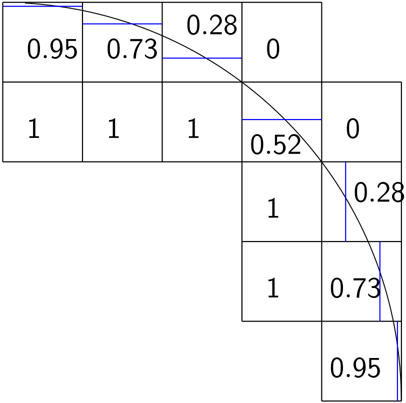
$$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c = 0.$$

Treat as a single 'effective' fluid

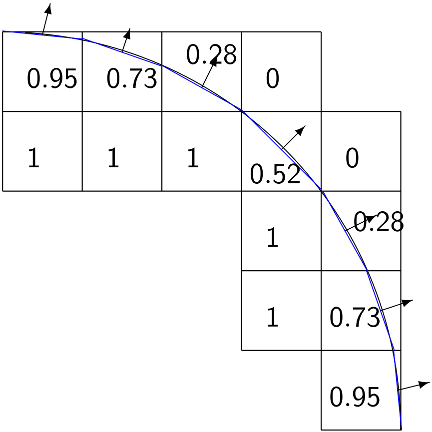
$$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c = 0.$$

Solve equations of motion in fixed Cartesian grid using conservative forms, with care in grid block containing interface

Reconstructing the interface from $c(\mathbf{x}, t)$



(a) SLIC – Simple Linear



(b) PLIC – Piecewise Linear

For capillary force need ∇c, but c discontinuous, so smooth/average

reconstructing the interface from $c(\mathbf{x}, t)$

Capillary force $\nabla\cdot\boldsymbol{\Sigma}$ in momentum conservation with

$$\boldsymbol{\Sigma} = \gamma(\mathbf{I} - \mathbf{nn})\delta(c - \tfrac{1}{2}) \qquad \mathbf{n} = \overline{\nabla c}/|\overline{\nabla c}|$$

Discontinuous c is unsuitable for numerical differentiation, so smooth/average

$$\left.\frac{\partial c}{\partial x}\right|_{i+\frac{1}{2}\,j+\frac{1}{2}} = \frac{1}{8\Delta x}\Big(c_{i+\frac{3}{2}\,j+\frac{3}{2}} + 2c_{i+\frac{3}{2}\,j+\frac{1}{2}} + c_{i+\frac{3}{2}\,j-\frac{1}{2}} \\ -c_{i-\frac{1}{2}\,j+\frac{3}{2}} - 2c_{i-\frac{1}{2}\,j+\frac{1}{2}} - c_{i-\frac{1}{2}\,j-\frac{1}{2}}\Big)$$

Diffuse Interface method

Phase-field function $\phi(\mathbf{x}, t)$

$$c(\mathbf{x}, t) = \begin{cases} 0 & \text{in fluid } 0, \\ 1 & \text{in fluid } 1 \end{cases}$$

with a transition layer between

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = \epsilon_1 \nabla^2 \phi - \frac{1}{\epsilon_2} \frac{d}{d\phi} \left(\phi^2 (1 - \phi)^2 \right).$$

Layer thickness $\sqrt{\epsilon_1 \epsilon_2}$ on time scale ϵ_2

Expensive to resolve fast thin layer which does not represent real physics

Level Sets

Really good idea: indicator function $\phi(\mathbf{x}, t)$ starts as roughly distance to initial surface (opposite signs on two sides), evolves

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = 0.$$

Hyperbolic, but ϕ smooth, so easier

Surface found at any time from

$$\phi(\mathbf{x}, t) = 0$$

Smooth ϕ gives precise position within a grid block

If contours of ϕ become crowded, restart.

Possible Fast Marching Method when interface velocity is determined locally (not fluid mechanics)