

Boundary Integral/Element Method

- ▶ For linear problems with known simple Greens functions
e.g. potential flows, Stokes flows
- ▶ Good for complex geometry
- ▶ Very good for free surface problems needing only \mathbf{u} on the surface

Laplace equation

$$\nabla^2 \phi = 0 \quad \text{in the volume } V$$
$$\phi \quad \text{or} \quad \frac{\partial \phi}{\partial n} \quad \text{given on the surface } S$$

where \mathbf{n} the unit normal to the surface out of the volume.

Need Greens function $G(\mathbf{x}, \boldsymbol{\xi})$, viewing $\boldsymbol{\xi}$ as a fixed parameter

$$\nabla_x^2 G = \delta(\mathbf{x} - \boldsymbol{\xi}) \quad \text{for } \mathbf{x} \text{ in } V$$

G need not satisfy any BC on S

∇_x means differentiate with respect to x

Greens identity (divergence theorem)

$$\int_S \left(\phi \frac{\partial G}{\partial n} - \frac{\partial \phi}{\partial n} G \right) dS(\mathbf{x}) = \int_V (\phi \nabla^2 G - \nabla^2 \phi G) dV(\mathbf{x})$$
$$= \int_V \phi(\mathbf{x}) \delta(\mathbf{x} - \boldsymbol{\xi}) dV(\mathbf{x})$$

$$= \phi(\boldsymbol{\xi}) \times \begin{cases} 0 & \text{if } \boldsymbol{\xi} \text{ outside } V, \\ 1 & \text{if } \boldsymbol{\xi} \text{ inside } V, \\ \frac{1}{2} & \text{if } \boldsymbol{\xi} \text{ on smooth } S, \\ \frac{1}{4}\Omega & \text{if } \boldsymbol{\xi} \text{ at corner of } S \text{ with solid angle } \Omega. \end{cases}$$

Boundary integral equation

For ξ on smooth S

$$\frac{1}{2}\phi(\xi) = \int_S \left(\phi \frac{\partial G}{\partial n} - \frac{\partial \phi}{\partial n} G \right) dS(\mathbf{x})$$

Either given $\phi|_S$, solve for $\frac{\partial \phi}{\partial n}|_S$

Or given $\frac{\partial \phi}{\partial n}|_S$, solve for $\phi|_S$

Then find ϕ inside V by evaluation integral with 1 replacing $\frac{1}{2}$

For exterior problem, add $\phi_\infty(\xi)$ to RHS of integral equation

Greens functions

Normally use 'free-space' Greens functions

$$\text{in } R^3: \quad G = -\frac{1}{4\pi|\mathbf{x} - \boldsymbol{\xi}|}, \quad \frac{\partial G}{\partial n} = \frac{(\mathbf{x} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x})}{4\pi|\mathbf{x} - \boldsymbol{\xi}|^3}$$

$$\text{and in } R^2: \quad G = \frac{1}{2\pi} \ln |\mathbf{x} - \boldsymbol{\xi}|, \quad \frac{\partial G}{\partial n} = \frac{(\mathbf{x} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x})}{2\pi|\mathbf{x} - \boldsymbol{\xi}|^2}$$

Become elliptic functions for axisymmetric

Sometimes use images so G satisfies BCs (simple geometries)

Eigensolutions

Interior problem has one eigensolution

$$\phi = 1 \quad \text{and} \quad \frac{\partial \phi}{\partial n} = 0 \quad \text{on } S$$

corresponding to

$$\phi(\mathbf{x}) = 1 \quad \text{in } V$$

Associated constraint

$$\int_S \frac{\partial \phi}{\partial n} dS = 0$$

from zero volume sources in $\nabla^2 \phi = 0$ in V .

Integrand is singular

For fixed ξ on S and \mathbf{x} moving on S

$$G \propto \frac{1}{|\mathbf{x} - \xi|} \quad \text{in } R^3, \quad G \propto \ln |\mathbf{x} - \xi| \quad \text{in } R^2.$$

Integrable but singular – take care numerically

On smooth S

$$\mathbf{n}(\mathbf{x}) \cdot (\mathbf{x} - \xi) \sim \frac{1}{2}\kappa |\mathbf{x} - \xi|^2,$$

where κ is the curvature. Hence

$$\frac{\partial G}{\partial n} \sim \frac{\kappa}{8\pi |\mathbf{x} - \xi|} \quad \text{in } R^3, \quad G \propto \frac{\kappa}{4\pi} \quad \text{in } R^2.$$

So no more singular

Hence need numerically smooth S

Discretise

- 1 Divided up S into 'panels'
in R^2 a curve divided into segments
in R^3 normally triangles
- 2 Represent unknowns ϕ and $\partial\phi/\partial n$ by basis functions $f_i(\mathbf{x})$ over the panels, e.g. piecewise constants/linear (or B -splines)

$$\phi(\mathbf{x}) = \sum \Phi_i f_i(\mathbf{x}), \quad \frac{\partial\phi}{\partial n} = \sum D\Phi_i f_i(\mathbf{x})$$

with unknown amplitudes Φ_i and $D\Phi_i$.

- 3 Satisfy integral equation at **collocation points**
or by least squares or with weighted integrals.

Suitable collocation points are:

centre of panels for piecewise constant basis functions

vertices of panels for piecewise linear basis functions.

Discretised integral equation

One thus forms a discretised version of the integral equation in terms of the amplitudes Φ_i and $D\Phi_i$

$$\left(\frac{1}{2}I - D\mathcal{G}\right)\Phi = -\mathcal{G}D\Phi,$$

where the matrix elements are

$$D\mathcal{G}_{ij} = \int_S f_j(\mathbf{x}) \frac{\partial G}{\partial n}(\mathbf{x}, \boldsymbol{\xi}) dS(\mathbf{x}), \quad \text{and} \quad \mathcal{G}_{ij} = \int_S f_j(\mathbf{x}) G(\mathbf{x}, \boldsymbol{\xi}) dS(\mathbf{x}),$$

both evaluated at $\boldsymbol{\xi} = \mathbf{x}_i$.

Evaluation of \mathcal{G} and $D\mathcal{G}$

Short range integrals (if splines must use B -splines)

Often use **Gaussian integration** – avoids singular point $\mathbf{x} = \boldsymbol{\xi}$

Often use **trapezoidal integration** for $|i - j| > 3$ or 4

Gaussian poor for self and next-to-self panels $|i - j| \leq 1$

8pt Gaussian \rightarrow error $3 \cdot 10^{-15}$ in $\int_0^\pi \sin x$, but $9 \cdot 10^{-3}$ in $\int_0^1 \ln x$

So **subtract off** the singularity and evaluate analytically

$G(x, \xi) \sim a(\xi) \ln |x - \xi| + \text{regular term.}$

$$\int_{\xi - \delta_1}^{\xi + \delta_2} a(\xi) \ln |x - \xi| dx = a(\xi) (\delta_2 \ln \delta_2 - \delta_2 + \delta_1 \ln \delta_1 - \delta_1).$$

Regular term safely by the trapezoidal rule.

Similarly the next-to-self panel, if not one more beyond.

Avoiding eigensolution

Invert singular matrices

$$\left(\frac{1}{2}I - D\mathcal{G}\right)\Phi = -\mathcal{G}D\Phi,$$

in space orthogonal to eigensolution

Fix 1 Rely on truncation error to keep condition number finite

Fix 2 Make eigenvalue α rather than 0

$$A' = A + \alpha ee^\dagger$$

For interior problem

$$e = (1, 1, \dots, 1) \quad \text{and} \quad (e^\dagger)_j = \int_S f_j dS$$

(so long as $\sum f_i(x) \equiv 1$)

Tests

In two dimensions

$$\phi = r^k \cos k\theta$$

$$\text{with } \frac{\partial \phi}{\partial n} = \mathbf{n} \cdot \nabla \phi = n_r k r^{k-1} \cos k\theta - n_\theta k r^{k-1} \sin k\theta,$$

and similarly in three dimensions.

Test error is $O(\Delta x^2)$ if piecewise linear basis functions,
and $O(\Delta x^4)$ if cubic splines

Costs

Boundary integral method has unknowns only on surface, so costs less?

- ▶ Volume method N^2 points in 2D, N^3 points in 3D
Fast Poisson solver (need regular geometry) $N \ln N$ steps
Cost $N^3 \ln N$ or $N^4 \ln N$
- ▶ Surface method $4N$ points in 2D, $6N^2$ points in 3D
Boundary integral method has **dense** matrix $\frac{1}{3}(\cdot)^3$ inversion
Costs $11N^3$ or $72N^6$

But BIM good for complex or ∞ geometry

Reduce cost to $(\cdot)^2$ by iteration from last time-step

Try Fast Multipoles

Free surface potential flows

Start time step with known surface $S(t)$ and potential $\phi(\mathbf{x}, t)$ known on S

Use BIM to find $\partial\phi/\partial n$ on S , $\rightarrow \nabla\phi$

Evolve surface

$$\frac{D\mathbf{x}}{Dt} = \nabla\phi \quad \text{for points on } S$$

Evolve surface potential

$$\frac{D\phi}{Dt} = \frac{1}{2}|\nabla\phi|^2 - \mathbf{g} \cdot \mathbf{x} - \frac{\gamma}{\rho}\kappa - p_{\text{atm}} \quad \text{for points } \mathbf{x} \text{ on } S,$$

Capillary waves mean $\Delta t < \sqrt{\rho/\gamma}\Delta x^{3/2}$

A good test is the vibration frequencies of an isolated drop.

Problem: conserve energy \rightarrow accumulate numerical noise in short capillary waves, so smooth or Fourier filter

Stokes flows

$$\frac{1}{2}\mathbf{u}(\boldsymbol{\xi}) = \int_S ((\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{G} - \mathbf{u} \cdot \mathbf{K} \cdot \mathbf{n}) dS(\mathbf{x}),$$

with the Greens function, called a Stokeslet, and its derivative

$$\mathbf{G} = \frac{1}{8\pi\mu} \left(\mathbf{I} \frac{1}{r} + \frac{\mathbf{r}\mathbf{r}}{r^3} \right) \quad \text{and} \quad \mathbf{K} = -\frac{3}{4\pi} \frac{\mathbf{r}\mathbf{r}\mathbf{r}}{r^5}, \quad \text{where} \quad \mathbf{r} = \mathbf{x} - \boldsymbol{\xi}.$$

For drops, outside minus inside, so only need $[\boldsymbol{\sigma} \cdot \mathbf{n}] = -\gamma\kappa\mathbf{n}$

$$\frac{1}{2}(\mu_{in} + \mu_{out})\mathbf{u}(\boldsymbol{\xi}) = \int_S ([\boldsymbol{\sigma} \cdot \mathbf{n}] \cdot \mathbf{G} - (\mu_{in} - \mu_{out})\mathbf{u} \cdot \mathbf{K} \cdot \mathbf{n}) dS(\mathbf{x}),$$

Eigensolutions of rigid body motion for interior problem – no motion from constant pressure