Part II continued – more details on general issues

Last time – Finite Differences

Higher orders – central, 1-sided, non-equispaced

Compact 4th order Poisson solver

Upwinding

Grids – non-Cartesian, stretched, staggered

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Finite Elements

Good for engineering problems with complex geometries

- 'just' need to triangulate domain

Good for elliptic, OK for parabolic, poor for hyperbolic

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Good for elliptic, OK for parabolic, poor for hyperbolic

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Poor difficult programming on unstructured grid

Poor no efficient Poisson solver on unstructured grid

Poor difficult presenting results on unstructured grid

Finite Elements = Two ideas

1. Simple representation for unknown function over the finite element
   - not point data of FD

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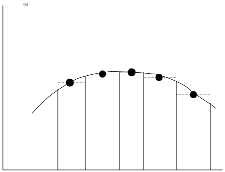
2. Weak formulation of the governing equations
   – variational statement

Representations in 1D

a. Constant elements

$$f(x) = f_i$$

in $x_{i-1} \leq x < x_i$



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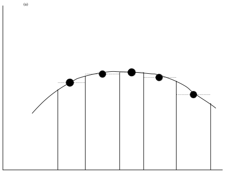

b. Linear elements

$$f(x) = f_{i-1}\frac{x_i - x}{x_i - x_{i-1}} + f_i\frac{x - x_{i-1}}{x_i - x_{i-1}}$$

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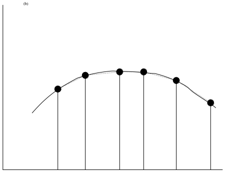

More representations in 1D

First map element to unit interval

$$x(\xi) = x_{i-1} + (x_i - x_{i-1})\xi \quad \text{for } 0 \leq \xi \leq 1$$

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$$f(x) = f_{i-1}(1-\xi)(1-2\xi) + f_{i-\frac{1}{2}} 4\xi(1-\xi) + f_i \xi(2\xi - 1)$$

NB: f' discontinuous at boundaries

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d. Cubic elements

Obvious generalisation, but better:

$$\begin{aligned} f(x) &= f_{i-1}(1-\xi)^2(1+2\xi) + f'_{i-1}(1-\xi)^2\xi \\ &\quad + f_i \xi^2(3-2\xi) + f'_i \xi^2(1-\xi), \end{aligned}$$

Now only f'' discontinuous at boundaries – see splines later

basis functions

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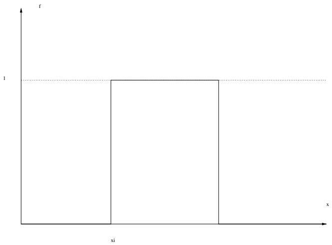
For the constant elements, the basis functions are

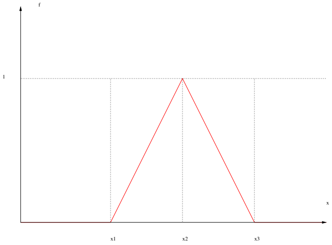
$$\phi_i(x) = \begin{cases} 1 & \text{in} \quad x_{i-1} \le x < x_i \\ 0 & \text{otherwise.} \end{cases}$$



Basis functions for linear elements

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & \text{in} \quad x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & \text{in} \quad x_i \le x \le x_{i+1} \\ 0 & \text{otherwise,} \end{cases}$$

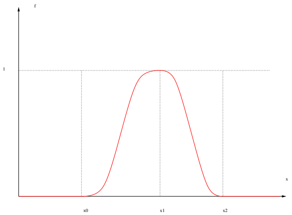
with obvious modifications for the end elements.

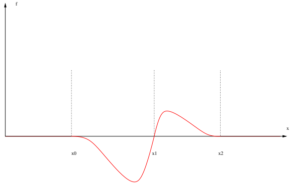


Basis functions for cubic elements

$$\phi_i(x) = \begin{cases} \frac{(x_{i+1}-x)^2(x_{i+1}+2x-3x_i)}{(x_{i+1}-x_i)^3} & \text{in} \quad x_i \le x < x_{i+1} \\ \frac{(x-x_{i-1})^2(3x_i-2x-x_{i-1})}{(x_i-x_{i-1})^3} & \text{in} \quad x_{i-1} \le x < x_i \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{\phi}_i(x) = \begin{cases} \frac{(x-x_i)(x_{i+1}-x)^2}{(x_{i+1}-x_i)^2} & \text{in} \quad x_i \le x < x_{i+1} \\ \frac{(x-x_i)(x-x_{i-1})^2}{(x_i-x_{i-1})^2} & \text{in} \quad x_{i-1} \le x < x_i \\ 0 & \text{otherwise.} \end{cases}$$





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Mostly triangles, sometimes rectangles

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$$\ell_{12}(x, y) = \frac{(x - x_1)(y_2 - y_1) - (x_2 - x_1)(y - y_1)}{(x_3 - x_1)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_1)}.$$

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Then

$$f(\mathbf{x}) = f_1\ell_{23}(\mathbf{x}) + f_2\ell_{31}(\mathbf{x}) + f_3\ell_{12}(\mathbf{x}).$$

Representation continuous over domain

more representations in 2D

c. Quadratic elements Values at vertices and mid-points

$$
\begin{aligned}
f(\mathbf{x}) \quad = \quad & f_1\ell_{23}(\mathbf{x})(2\ell_{23}(\mathbf{x})-1) \\
& +f_2\ell_{31}(\mathbf{x})(2\ell_{31}(\mathbf{x})-1) \\
& +f_3\ell_{12}(\mathbf{x})(2\ell_{12}(\mathbf{x})-1) \\
& +f_{23}4\ell_{12}(\mathbf{x})\ell_{31}(\mathbf{x})+f_{31}4\ell_{23}(\mathbf{x})\ell_{12}(\mathbf{x})+f_{12}4\ell_{31}(\mathbf{x})\ell_{23}(\mathbf{x}).
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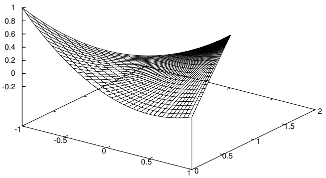

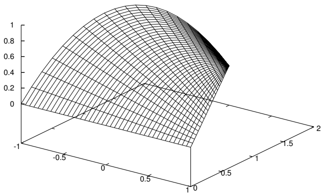

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Local nature $\rightarrow$ sparse coupling matrices for PDEs

more representations in 2D

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more representations in 2D

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Bilinear, taking values at vertices

$$f(\mathbf{x}) = f_1\xi\eta + f_2(1-\xi)\eta + f_3\xi(1-\eta) + f_4(1-\xi)(1-\eta).$$

Continuous over domain.

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Continuous and continuous tangential derivative at boundaries.

Variational statement of Poisson problem

$$\nabla^2 f = \rho \quad \text{in volume } V$$

$$\text{with boundary condition, say} \quad f = g \quad \text{on surface } S,$$

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Rayleigh-Ritz variational formulation: out of all those functions $f(\mathbf{x})$ that satisfy BCs, the one that minimises

$$I(f) = \int_V \left(\frac{1}{2} |\nabla f|^2 + \rho f \right) dV$$

also satisfies the Poisson problem.

Substitute FE representation

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The weak formulation of the PDE (f can be non-C^2)

Details in 1D

$$\frac{d^2 f}{dx^2} = \rho \quad \text{in } a < x < b, \quad \text{with } f(a) = A \text{ and } f(b) = B,$$

where $\rho(x), A$ and B given.

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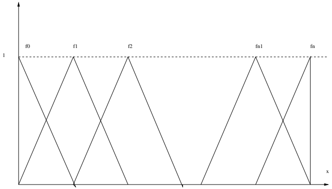

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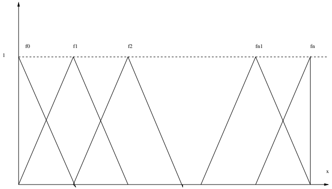

Unknown $f(x)$ represented (BCs built in)

$$f(x) = A\phi_0(x) + B\phi_N(x) + \sum_{i=1}^{N-1} f_i \phi_i(x)$$

more details in 1D

At interior pts

$$K_{ij} = \int \nabla\phi_i \cdot \nabla\phi_j = \begin{cases} 2/h & \text{if } i = j, \\ -1/h & \text{if } i = j \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

by $\nabla\phi_i = 0, +1, -1, 0$

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– same for the point values in the finite difference approach.

more details in 1D

Remark If evaluate r_i more accurately

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So obtain f_i to $O(h^4)$.

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i.e. FE approach naturally conservative.