Last time

- Accuracy – first- and second-order
- Stability – CFL condition
- Longwaves – diffusion (odd), dispersion (even)

- Simplest – unstable
- Lax Friedricks – too stable, first-order, diffusion
- Upwinding – stable, first-order, diffusion
- Crank-Nicolson, second-order, implicit, dispersion
- Lax Wendroff – second-order, explicit, dispersion
- Angled second-order, explicit, dispersion

2. Simple advection of unsmooth ICs

$$u_t + cu_x = 0, \quad c > 0, \text{ const}$$

with discontinuous initial conditions

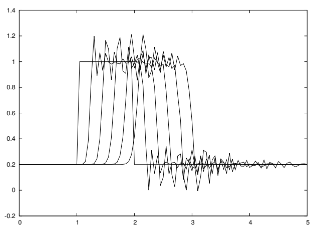
$$u = \begin{cases} 1 & 2 \leq x \leq 3 \\ 0.2 & \text{otherwise} \end{cases}$$

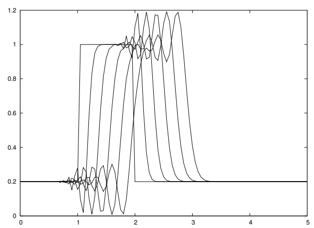
Problems with errors $\Delta x^2 u_{xxx}$ when $u_x = \infty$

simple advection of unsmooth ICs

High-order schemes give spurious oscillations

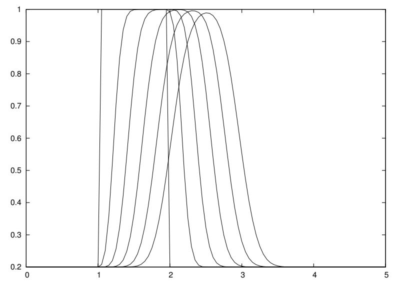
Angled Derivative and Lax-Wendroff





simple advection of unsmooth ICs

Upwinding algorithm



No oscillations but lots of damping

Need new idea

3. Total Variation Diminishing

$$TV(u^n) = \sum_\ell |u^n_{\ell+1} - u^n_\ell|.$$

i.e. sum of all the differences between adjacent minima and maxima, so independent of numerical resolution.

A TVD algorithm: total variation does not increase in time

$$TV(u^{n+1}) \le TV(u^n).$$

No spurious oscillations with new minima and maxima.

Preserves the monotonicity of a section of the solution.

Flux-limiters

Idea: method = low-order (Upwind) + high-order correction (LxW)
Limiter 1: switch off correction in oscillation
Limiter 2: reduce correction if gradient changes rapidly

First reformulated in conservation form with divergence of fluxes f

$$u^{n+1}_\ell = u^n_\ell - \frac{\Delta t}{\Delta x}\left(f^n_{\ell+\frac{1}{2}} - f^n_{\ell-\frac{1}{2}}\right).$$

For Upwinding plus Lax-Wendroff correction (for $c > 0$)

$$f^n_{\ell+\frac{1}{2}} = cu^n_\ell + \tfrac{1}{2}c(\Delta x - c\Delta t)u'_{\ell+\frac{1}{2}},$$

where $\quad u'_{\ell+\frac{1}{2}} = \dfrac{u^n_{\ell+1} - u^n_\ell}{\Delta x}$ to be limited by the upstream $\quad \dfrac{u^n_\ell - u^n_{\ell-1}}{\Delta x}$.

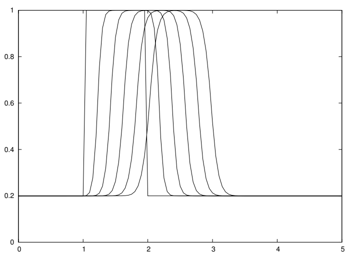
(If $c < 0$, the upstream side switches)

eg flux-limiters – Minmod

$$a = \frac{u^n_{\ell+1} - u^n_\ell}{\Delta x} \quad \text{is to be limited by} \quad b = \frac{u^n_\ell - u^n_{\ell-1}}{\Delta x}$$

$$\mathrm{Minmod}(a,b) = \begin{cases} 0 & \text{if } ab < 0 \\ a & \text{if } ab > 0 \text{ and } |a| < |b| \\ b & \text{if } ab > 0 \text{ and } |b| < |a| \end{cases}$$

i.e. 0 in oscillation and smaller slope if monotone.

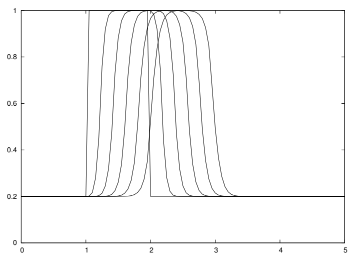


$ct = 0.0\,(0.2)\,1.0$
$\Delta x = 0.05$ and $c\Delta t = 0.0125$

eg flux-limiters – Superbee

$$\mathrm{Superbee}(a,b) = \begin{cases} 0 & \text{if } ab < 0, \\ a & \text{if } ab > 0 \text{ and } \left(|a| < \frac{1}{2}|b| \text{ or } |b| < |a| < 2|b|\right), \\ b & \text{if } ab > 0 \text{ and } \left(|b| < \frac{1}{2}|a| \text{ or } |a| < |b| < 2|a|\right) \end{cases}$$

i.e. 0 in oscillation and when monotone larger if less than twice smaller, otherwise smaller.



$ct = 0.0\,(0.2)\,1.0$
$\Delta x = 0.05$ and $c\Delta t = 0.0125$
slightly sharper than Minmod

4. Nonlinear advection

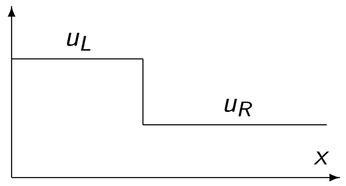
Conservative form

$$u_t + (f(u))_x = 0$$

Propagation form

$$u_t + f'(u)u_x = 0$$

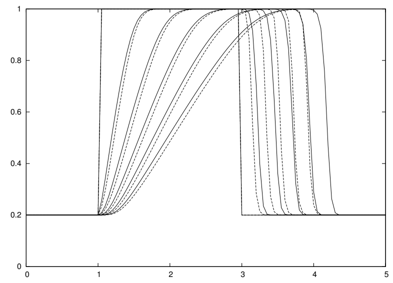
Possibility of shockwaves e.g. $f'(u) > 0$ when $u_x < 0$.



shock speed $V = \dfrac{f(u_L) - f(u_R)}{u_L - u_L}$

Conservative scheme gives correct shock speed

Case of flux $f(u) = \frac{1}{2}u^2$. Shock speed $V = 0.6$
Upwinding, $\Delta x = 0.05$, $\Delta t = 0.0125$, $ct = 0.0\,(0.4)\,2.0$



Continuous curve conservative scheme, $V = 0.59$
Dashed curve propagation scheme, $V = 0.46$

Godunov method

Three steps

R. Reconstruct the solution into a simple form.
Normally a constant in each grid block, occasionally linear.
Note the discontinuities at the boundaries of the grid blocks.

E. The simple form is evolved exactly.
Constant parts are advected at a constant speed.
The discontinuities are propagated as shockwaves or rarefaction waves.
The time-step must be limited by the CFL condition to stop discontinuities propagating through more than one grid block.

A. The resulting function is averaged over grid blocks in preparation for step R of the next time-step.

Skip second step by using fluxes from upstream side, but which is upstream?

Godunov - upstream fluxes

For general flux $f(u)$, information propagates at $f'(u)$.
Flux $f_{\ell+\frac{1}{2}}$ from grid block ℓ to grid block $\ell + 1$

$$f_{\ell+\frac{1}{2}} = \begin{cases} f(u_\ell) & \text{if } f'(u_\ell) > 0, f'(u_{\ell+1}) > 0, \\ f(u_{\ell+1}) & \text{if } f'(u_\ell) < 0, f'(u_{\ell+1}) < 0, \\ \\ f(u_\ell) & \text{if } f'(u_\ell) > 0, f'(u_{\ell+1}) < 0, V > 0, \\ f(u_{\ell+1}) & \text{if } f'(u_\ell) > 0, f'(u_{\ell+1}) < 0, V < 0, \\ \\ f(u_*) & \text{if } f'(u_\ell) < 0, f'(u_{\ell+1}) > 0, \text{ where, } f'(u_*) = 0. \end{cases}$$

Last case rarefaction wave, so flux for value u_* which does not propagate
shock speed $V = \frac{f(u_L) - f(u_R)}{u_L - u_L}$

Further

Godonov to higher orders

Godonov is first-order in time and space, through the averaging, with large numerical diffusion $O(\Delta x^2/\Delta t)$

Replace piecewise constant by piecewise linear. Slopes can be flux-limited.

Extension to systems $\mathbf{u}(x,t)$.

– diagonalise $\mathbf{f}'(\mathbf{u})$ to find what information propagates in what direction.

Higher dimensions $u(\mathbf{x},t)$

– Riemann solvers do not work

Finite Elements

– distribute fluxes over vertices