Boundary Integral/Element Method

- For linear problems with known simple Greens functions

  e.g. potential flows, Stokes flows

- Good for complex geometry

- Very good for free surface problems needing only $\mathbf{u}$ on the surface

Greens identity (divergence theorem)

$$\int_S \left(\phi \frac{\partial G}{\partial n} - \frac{\partial \phi}{\partial n} G \right) \, dS(\mathbf{x}) = \int_V \left(\phi \nabla^2 G - \nabla^2 \phi \, G \right) \, dV(\mathbf{x})$$

$$= \int_V \phi(\mathbf{x}) \delta(\mathbf{x} - \boldsymbol{\xi}) \, dV(\mathbf{x})$$

$$= \phi(\boldsymbol{\xi}) \times \begin{cases} 0 & \text{if } \boldsymbol{\xi} \text{ outside } V, \\ 1 & \text{if } \boldsymbol{\xi} \text{ inside } V, \\ \frac{1}{2} & \text{if } \boldsymbol{\xi} \text{ on smooth } S, \\ \frac{1}{4}\Omega & \text{if } \boldsymbol{\xi} \text{ at corner of } S \text{ with solid angle } \Omega. \end{cases}$$

Laplace equation

$$\nabla^2 \phi = 0 \quad \text{in the volume } V$$

$$\phi \quad \text{or} \quad \frac{\partial \phi}{\partial n} \quad \text{given on the surface } S$$

where $\mathbf{n}$ the unit normal to the surface out of the volume.

Need Greens function $G(\mathbf{x}, \boldsymbol{\xi})$, viewing $\boldsymbol{\xi}$ as a fixed parameter

$$\nabla_x^2 G = \delta(\mathbf{x} - \boldsymbol{\xi}) \quad \text{for } \mathbf{x} \text{ in } V$$

G need not satisfy any BC on S

∇_x means differentiate with respect to x

Boundary integral equation

For ξ on smooth S

$$\tfrac{1}{2}\phi(\boldsymbol{\xi}) = \int_S \left(\phi \frac{\partial G}{\partial n} - \frac{\partial \phi}{\partial n} G \right) \, dS(\mathbf{x})$$

Either given $\phi|_S$, solve for $\frac{\partial \phi}{\partial n}|_S$

Or given $\frac{\partial \phi}{\partial n}|_S$, solve for $\phi|_S$

Then find ϕ inside V by evaluation integral with 1 replacing $\frac{1}{2}$

For exterior problem, add $\phi_\infty(\xi)$ to RHS of integral equation

Greens functions

Normally use 'free-space' Greens functions

$$\text{in } R^3: \quad G = -\frac{1}{4\pi|\mathbf{x}-\boldsymbol{\xi}|}, \quad \frac{\partial G}{\partial n} = \frac{(\mathbf{x}-\boldsymbol{\xi})\cdot\mathbf{n}(\mathbf{x})}{4\pi|\mathbf{x}-\boldsymbol{\xi}|^3}$$

$$\text{and in } R^2: \quad G = \frac{1}{2\pi}\ln|\mathbf{x}-\boldsymbol{\xi}|, \quad \frac{\partial G}{\partial n} = \frac{(\mathbf{x}-\boldsymbol{\xi})\cdot\mathbf{n}(\mathbf{x})}{2\pi|\mathbf{x}-\boldsymbol{\xi}|^2}$$

Become elliptic functions for axisymmetric

Sometimes use images so G satisfies BCs (simple geometries)

Eigensolutions

Interior problem has one eigensolution

$$\phi = 1 \quad \text{and} \quad \frac{\partial\phi}{\partial n} = 0 \quad \text{on } S$$

corresponding to

$$\phi(\mathbf{x}) = 1 \quad \text{in } V$$

Associated constraint

$$\int_S \frac{\partial\phi}{\partial n}\,dS = 0$$

from zero volume sources in $\nabla^2\phi = 0$ in V.

Integrand is singular

For fixed $\boldsymbol{\xi}$ on S and $\mathbf{x}$ moving on S

$$G \propto \frac{1}{|\mathbf{x}-\boldsymbol{\xi}|} \quad \text{in } R^3, \qquad G \propto \ln|\mathbf{x}-\boldsymbol{\xi}| \quad \text{in } R^2.$$

Integrable but singular – take care numerically

On smooth S

$$\mathbf{n}(\mathbf{x})\cdot(\mathbf{x}-\boldsymbol{\xi}) \sim \tfrac{1}{2}\kappa|\mathbf{x}-\boldsymbol{\xi}|^2,$$

where κ is the curvature. Hence

$$\frac{\partial G}{\partial n} \sim \frac{\kappa}{8\pi|\mathbf{x}-\boldsymbol{\xi}|} \quad \text{in } R^3, \qquad G \propto \frac{\kappa}{4\pi} \quad \text{in } R^2.$$

So no more singular
Hence need numerically smooth S

Discretise

1. Divided up S into 'panels'
   in R^2 a curve divided into segments
   in R^3 noramlly triangles
2. Represnt unknowns ϕ and $\partial\phi/\partial n$ by basis functions $f_i(\mathbf{x})$ over the panels, e.g. piecewise constants/linear (or B-splines)

   $$\phi(\mathbf{x}) = \sum \Phi_i f_i(\mathbf{x}), \quad \frac{\partial\phi}{\partial n} = \sum D\Phi_i f_i(\mathbf{x})$$

   with unknown amplitudes Φ_i and $D\Phi_i$.
3. Satisfy integral equation at collocation points
   or by least squares or with weighted integrals.

Suitable collocations points are:
centre of panels for piecewise constant basis functions
vertices of panels for piecewise linear basis functions.

Discretised integral equation

One thus forms a discretised version of the integral equation in terms of the amplitudes Φ_i and $D\Phi_i$

$$\left(\tfrac{1}{2}I - D\mathcal{G}\right)\Phi = -\mathcal{G}D\Phi,$$

where the matrix elements are

$$D\mathcal{G}_{ij} = \int_S f_j(\mathbf{x})\frac{\partial G}{\partial n}(\mathbf{x},\boldsymbol{\xi})\,dS(\mathbf{x}), \quad \text{and} \quad \mathcal{G}_{ij} = \int_S f_j(\mathbf{x})G(\mathbf{x},\boldsymbol{\xi})\,dS(\mathbf{x}),$$

both evaluated at $\boldsymbol{\xi} = \mathbf{x}_i$.

Evaluation of $\mathcal{G}$ and $D\mathcal{G}$

Short range integrals (if splines must use B-splines)

Often use Gaussian integration – avoids singular point $\mathbf{x} = \boldsymbol{\xi}$

Often use trapezoidal integration for $|i-j| > 3$ or 4

Gaussian poor for self and next-to-self panels $|i-j| \le 1$

8pt Gaussian $\to$ error $3\,10^{-15}$ in $\int_0^\pi \sin x$, but $9\,10^{-3}$ in $\int_0^1 \ln x$

So subtract off the singularity and evaluate analytically

$$G(x,\xi) \sim a(\xi)\ln|x-\xi| + \text{regular term.}$$

$$\int_{\xi-\delta_1}^{\xi+\delta_2} a(\xi)\ln|x-\xi|\,dx = a(\xi)\left(\delta_2\ln\delta_2 - \delta_2 + \delta_1\ln\delta_1 - \delta_1\right).$$

Regular term safely by the trapezoidal rule.
Similarly the next-to-self panel, if not one more beyond.

Avoiding eigensolution

Invert singular matrices

$$\left(\tfrac{1}{2}I - D\mathcal{G}\right)\Phi = -\mathcal{G}D\Phi,$$

in space orthogonal to eigensolution

Fix 1 Rely on truncation error to keep condition number finite

Fix 2 Make eigenvlue α rather than 0

$$A' = A + \alpha e e^\dagger$$

For interior problem

$$e = (1,1,\ldots,1) \quad \text{and} \quad \left(e^\dagger\right)_j = \int_S f_j\,dS$$

(so long as $\sum f_i(x) \equiv 1$)

Tests

In two dimensions

$$\phi = r^k\cos k\theta$$

$$\text{with} \quad \frac{\partial\phi}{\partial n} = \mathbf{n}\cdot\nabla\phi = n_r k r^{k-1}\cos k\theta - n_\theta k r^{k-1}\sin k\theta,$$

and similarly in three dimensions.

Test error is $O(\Delta x^2)$ if piecewise linear bassis functions,
and $O(\Delta x^4)$ if cubic splines

Costs

Boundary integral method has unknowns only on surface, so costs less?

- Volume method N^2 points in 2D, N^3 points in 3D
  Fast Poisson solver (need regular geometry) $N \ln N$ steps
  Cost $N^3 \ln N$ or $N^4 \ln N$
- Surface method $4N$ points in 2D, $6N^2$ points in 3D
  Boundary integral method has dense matirx $\frac{1}{3}(.)^3$ inversion
  Costs $11N^3$ or $72N^6$

But BIM good for complex or ∞ geometry

Reduce cost to $(.)^2$ by interation from last time-step

Try Fast Multipoles

Stokes flows

$$\tfrac{1}{2}\mathbf{u}(\boldsymbol{\xi}) = \int_S \left((\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\mathbf{G} - \mathbf{u}\cdot\mathbf{K}\cdot\mathbf{n}\right) dS(\mathbf{x}),$$

with the Greeens function, called a Stokeslet, and its derivative

$$\mathbf{G} = \frac{1}{8\pi\mu}\left(\mathbf{I}\frac{1}{r} + \frac{\mathbf{r}\,\mathbf{r}}{r^3}\right) \quad \text{and} \quad \mathbf{K} = -\frac{3}{4\pi}\frac{\mathbf{r}\,\mathbf{r}\,\mathbf{r}}{r^5}, \quad \text{where} \quad \mathbf{r} = \mathbf{x} - \boldsymbol{\xi}.$$

For drops, outside minus inside, so only need $[\boldsymbol{\sigma}\cdot\mathbf{n}] = -\gamma\kappa\mathbf{n}$

$$\tfrac{1}{2}(\mu_{in} + \mu_{out})\mathbf{u}(\boldsymbol{\xi}) = \int_S \left([\boldsymbol{\sigma}\cdot\mathbf{n}]\cdot\mathbf{G} - (\mu_{in} - \mu_{out})\mathbf{u}\cdot\mathbf{K}\cdot\mathbf{n}\right) dS(\mathbf{x}),$$

Eigensolutions of rigid body motion for interior problem – no motion from constant pressure

Free surface potential flows

Start time step with known surface $S(t)$ and potential $\phi(\mathbf{x},t)$ known on S
Use BIM to find $\partial\phi/\partial n$ on S, $\to \nabla\phi$
Evolve surface

$$\frac{D\mathbf{x}}{Dt} = \nabla\phi \quad \text{for points on } S$$

Evolve surface potential

$$\frac{D\phi}{Dt} = \tfrac{1}{2}|\nabla\phi|^2 - \mathbf{g}\cdot\mathbf{x} - \frac{\gamma}{\rho}\kappa - p_{\mathrm{atm}} \quad \text{for points } \mathbf{x} \text{ on } S,$$

Capillary waves mean $\Delta t < \sqrt{\rho/\gamma}\,\Delta x^{3/2}$

A good test is the vibration frequencies of an isolated drop.

Problem: conserve energy $\to$ accumulate numerical noise in short capillary waves, so smooth or Fourier filter