Wavelets

- Compress audio signals and images
- Reveal structure in turbulence
  but yet to give econmical algorithm for turbulence

- Local Finite Differences
  – good for discontinuities
  – poor for waves, 8+ points per cycle
- Global Spectral
  – good for waves
  – poor for discontinuities, $\tilde{f} \sim 1/k$ with no wave of period $2\pi/k$ (NB $k^{-5/3}$ spectrum of turbulence)

Wavelets: best of both: local waves

Musical tune: sequence of notes of different amplitude, frequency, duration
Fourier not see finite duration, FD need 8+ points per cycle
Musical score very economical $\to$ wavelets

Possible wavelets

Morlet

$$(e^{ikx} - e^{-k^2/2})e^{-x^2/2}$$

Mexican hat (Marr)

$$\frac{d^2}{dx^2}\left(e^{-x^2/2}\right) = (x^2 - 1)e^{-x^2/2}$$

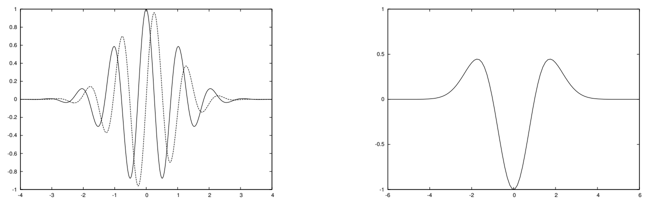


Figure : (a) Morlet wavelet with $k = 6$. (b) Marr Mexican hat wavelet.

Decay rapidly in x annd Fourier k

Continuous Wavelet Transform

Mother wavelet $\psi(x)$: translate through b, dilate by a

$$\psi_{a,b} = a^{-1/2}\psi\left(\frac{x-b}{a}\right)$$

Wavelet components

$$f_{a,b} = \int \psi^*_{a,b}(x) f(x)\, dx$$

Invert

$$f(x) = \frac{1}{C_\psi}\int f_{a,b}\psi_{a,b}\frac{da\,db}{a^2} \qquad \text{where } C_\psi = \int \frac{|\tilde{\psi}(s)|^2}{|k|}\,dk$$

For PDEs: $\frac{\partial}{\partial b} f_{a,b} = \left(\frac{\partial f}{\partial x}\right)_{a,b}$

Discrete Wavelet Transform

For unit interval $[0,1]$, with periodic extension (on a circle)
$N = 2^n$ points, $\quad x_k = k/N$ for $k = 0, 1, \dots, N-1$

Restrict to discrete set of translations and dilations

$$\psi_{i,j} = 2^{i/2}\psi(2^i x - j)$$

for $i = 0, \dots, n-1$ and $j = 0, \dots 2^i - 1$

E.g. one wavelet $\psi_{0,0} = \psi(x)$ on $[0,1]$
Two $\psi_{1,0} = \sqrt{2}\psi(2x)$ on $[0, \frac{1}{2}]$ and $\psi_{1,1} = \sqrt{2}\psi(2x-1)$ on $[\frac{1}{2}, 1]$
Down to finest level with 2^{n-1} wavelets on 2^{n-1} subintervals
Total number of wavelets $= 1 + 2 + \dots 2^{n-1} = N$ on N data points

A Multiscale representation

... Discrete Wavelet Transform

Wavelet components (using periodic extension near boundary)

$$f_{i,j} = \frac{1}{N}\sum_k \psi_{i,j}(x_k) f(x_k)$$

If $\psi(x)$ is nonzero only on unit interval,
then $f_{0,0}$ is sum over N points,
$f_{1,0}$ and $f_{1,1}$ are each sums over $N/2$ points, etc
Hence cost of all components is $O(N \ln_2 N)$

Advantage of special orthogonal wavelets (discrete)

$$\frac{1}{N}\sum_k \psi_{i,j}(x_k)\psi_{l,m}(x_k) = \delta_{il}\delta_{jm}$$

Then inverse discete wavelet transform

$$f(x_k) = \sum_{i,j} f_{i,j}\psi_{i,j}(x_k)$$

Possible orthogonal wavelets: Haar, Sinc, Meyer, Battle-Lemarié, Daubechies, symlets, Coiflets

Fast Wavelet Transform – $O(N)$

Start with simple case of Haar wavelet

$$\psi(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < \frac{1}{2}, \\ -1 & \text{if} \quad \frac{1}{2} \le x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Non-zero on single interval, so obviously orthogonal

Consider simple case of $N = 8 = 2^3$ points. Wavelet components

$$f_{0,0} = \tfrac{1}{8}(f_0 + f_1 + f_2 + f_3 - f_4 - f_5 - f_6 - f_7),$$
$$f_{1,0} = \tfrac{\sqrt{2}}{8}(f_0 + f_1 - f_2 - f_3), \quad f_{1,1} = \tfrac{\sqrt{2}}{8}(f_4 + f_5 - f_6 - f_7),$$
$$f_{2,0} = \tfrac{1}{4}(f_0 - f_1), \quad f_{2,1} = \tfrac{1}{4}(f_2 - f_3),$$
$$f_{2,2} = \tfrac{1}{4}(f_4 - f_5), \quad f_{2,3} = \tfrac{1}{4}(f_6 - f_7).$$

Problem 1: mean value. Problem 2: duplication

... Fast Wavelet Transform

The 7 components cannot represent 8 data points. Missing mean value, so introduce

$$f^{\phi}_{0,0} = \tfrac{1}{8}(f_0 + f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7).$$

Then inversion works, e.g. (student exercise!)

$$f_0 = f^{\phi}_{0,0}\phi_{0,0}(0) + f_{0,0}\psi_{0,0}(0) + f_{1,0}\psi_{1,0}(0) + f_{2,0}\psi_{2,0}(0),$$

with $\phi_{0,0}(0) = \psi_{0,0}(0) = 1, \quad \psi_{1,0}(0) = \sqrt{2}$ and $\psi_{2,0}(0) = 2.$

... Fast Wavelet Transform

Need scaling function $\phi(x)$, which for Haar is

$$\phi(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Same dilations and translations of this basic scaling function

$$\phi_{i,j}(x) = 2^{i/2}\phi(2^i x - j)$$

for $i = 0, \ldots, n-1$ and $j = 0, \ldots, 2^i - 1$
Similar components

$$f^{\phi}_{i,j} = \frac{1}{N}\sum_k \phi_{i,j}(x_k) f(x_k).$$

... Fast Wavelet Transform

The Fast transform: Start at finest level

$$f_{2,0} = \tfrac{1}{4}(f_0 - f_1) \quad f^{\phi}_{2,0} = \tfrac{1}{4}(f_0 + f_1)$$

and similarly other $f_{2,j}$ $f^{\phi}_{2,j}$
Next level up

$$f_{1,0} = \tfrac{1}{\sqrt{2}}(f^{\phi}_{2,0} - f^{\phi}_{2,1}) \quad f^{\phi}_{1,0} = \tfrac{1}{\sqrt{2}}(f^{\phi}_{2,0} + f^{\phi}_{2,1}),$$

Similarly next and coarsest level

$$f_{0,0} = \tfrac{1}{\sqrt{2}}(f^{\phi}_{1,0} - f^{\phi}_{1,1}) \quad f^{\phi}_{0,0} = \tfrac{1}{\sqrt{2}}(f^{\phi}_{1,0} + f^{\phi}_{1,1}).$$

Cost: $4N$ operations

... Fast Wavelet Transform

The Inverse Transform: have $f^{\phi}_{0,0}$ and all wavelets $f_{i,j}$
Start at coarsest level

$$f^{\phi}_{1,0} = \tfrac{1}{\sqrt{2}}(f^{\phi}_{0,0} + f_{0,0}) \quad f^{\phi}_{1,1} = \tfrac{1}{\sqrt{2}}(f^{\phi}_{0,0} - f_{0,0})$$

Similary generate all $f^{\phi}_{i,j}$ from coarser level
Finally recover the data

$$f_0 = \tfrac{1}{2}(f^{\phi}_{2,0} + f_{2,0}) \quad f_1 = \tfrac{1}{2}(f^{\phi}_{2,0} - f_{2,0})$$

and similarly all other f_k

NB: The Fast Transform and its Inverse do not use the values of the function, just the filter coefficients $\pm\frac{1}{\sqrt{2}}$
Gives generalisation from Haar to other orthogonal wavelets

... Fast Wavelet Transform

At any lth stage, the partial sum

$$\sum_{j=0}^{2^l-1} f^{\phi}_{l,j}\phi_{l,j}(x)$$

represents all the courser scale variations of the function which have not been described by wavelets at the scale of l and finer, as in

$$\sum_{j,i\geq l} f_{i,j}\psi_{i,j}(x).$$

Fast Wavelet Transform is a bank of frequency filters in signal processing
– a high-pass to the wavelet components and a low-pass to the remaining scaling compnents

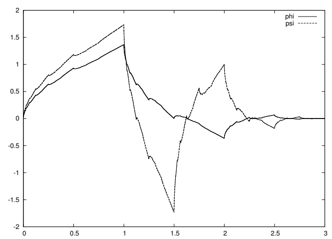
Daubechies Wavelets

A scaling function must be a linear combination of finer scale scaling functions. The Daubechies D-2 has just four, so that

$$\phi(x) = \sqrt{2}\,(h_o\phi(2x) + h_1\phi(2x-1) + h_2\phi(2x-2) + h_3\phi(2x-3))$$

Constraints of orthogonality, normalisation and some vanishing moments require

$$h_o = \frac{1+\sqrt{3}}{4\sqrt{2}}, \quad h_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}, \quad h_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}, \quad h_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$$



Distinctly irregular
Not good for PDEs
But just use filter coefficients

..Daubechies Wavelets

The Fast D-2 Wavelet Transform

$$f^{\phi}_{i,j} = \sum_k h_k f^{\phi}_{i+1,2i+k} \quad f_{i,j} = \sum_k g_k f^{\phi}_{i+1,2i+k},$$

where

$$g_0 = -h_3 \quad g_1 = h_2 \quad g_2 = -h_1 \quad g_3 = h_0$$

There is good Wavelet Toolbox in MATLAB