Mathematical Tripos Part IB — E.J. Hinch
FLUID DYNAMICS — October 2004

Example Sheet 1

1. Consider the two-dimensional flow $u = 1/(1+t)$, $v = 1$ in $t > -1$. Find and sketch
 (i) the streamline at $t = 0$ that passes through the point $(1, 1)$,
 (ii) the path of a fluid particle released from $(1, 1)$ at $t = 0$,
 (iii) the position at $t = 0$ of a streak of dye released from $(1, 1)$ during the time interval $-1 < t \leq 0$.

2. A steady two-dimensional flow (pure straining) is given by $u = \alpha x$, $v = -\alpha y$ with α constant.
 (i) Find the equation for a general streamline of the flow, and sketch some of them.
 (ii) At $t = 0$ the fluid on the curve $x^2 + y^2 = a^2$ is marked (by an electro-chemical technique). Find the equation for this material fluid curve for $t > 0$.
 (iii) Does the area within the curve change in time, and why?

3. Repeat question 2 for the two-dimensional flow (simple shear) given by $u = \gamma y$, $v = 0$ with γ constant. Which of the two flows stretches the curve faster at long times?

4. A two-dimensional flow is represented by a streamfunction $\psi(x, y)$ with $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Show that
 (i) the streamlines are given by $\psi =$ const,
 (ii) $|\mathbf{u}| = |\nabla\psi|$, so that the flow is faster where the streamlines are closer,
 (iii) the volume flux per unit z-distance crossing any curve from $\mathbf{x}_0$ to $\mathbf{x}_1$ is given by $\psi(\mathbf{x}_1) - \psi(\mathbf{x}_0)$,
 (iv) $\psi =$ const on any *fixed* (i.e. stationary) boundary.
 [*Hint for (iii):* $\mathbf{n}\, ds = (dy, -dx)$.]

5. Verify that the two-dimensional flow given in Cartesian coordinates by

$$u = \frac{y-b}{(x-a)^2 + (y-b)^2}, \qquad v = \frac{a-x}{(x-a)^2 + (y-b)^2}$$

satisfies $\nabla\cdot\mathbf{u} = 0$, and then find the streamfunction $\psi(x, y)$ such that $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Sketch the streamlines.

6. Verify that the two-dimensional flow given in polar coordinates by

$$u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \qquad u_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

satisfies $\nabla\cdot\mathbf{u} = 0$, and find the streamfunction $\psi(r, \theta)$. Sketch the streamlines.

$$\left[\textit{Take:} \quad \nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(u_\theta) \quad \text{and} \quad u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = -\frac{\partial\psi}{\partial r}.\right]$$

7. Verify that the axisymmetric flow (uniaxial straining) given in cylindrical polar coordinates by $u_r = -\frac{1}{2}\alpha r$, $u_z = \alpha z$ satisfies $\nabla\cdot\mathbf{u} = 0$, and find the Stokes streamfunction $\Psi(r, z)$. Sketch the streamlines.

$$\left[\textit{Take:} \qquad \nabla\cdot\mathbf{u} \;=\; \frac{1}{r}\frac{\partial}{\partial r}(ru_r) \;+\; \frac{\partial u_z}{\partial z} \quad \text{and} \quad u_r \;=\; -\frac{1}{r}\frac{\partial \Psi}{\partial z}, \quad u_z \;=\; \frac{1}{r}\frac{\partial \Psi}{\partial r}. \right]$$

8. An axisymmetric jet of water of speed $1\,\mathrm{m\,s^{-1}}$ and cross-section $6\times 10^{-4}\,\mathrm{m}^2$ strikes a wall at right angles and spreads out over it. By using the momentum integral equation, calculate the force on the wall due to the jet. [Neglect gravity.]

9. Starting from the Euler momentum equation for a fluid of constant density with a potential force $-\nabla\Phi$ per unit mass, show that for fixed volume V enclosed by surface A

$$\frac{d}{dt}\int_V \tfrac{1}{2}\rho|\mathbf{u}|^2\,dV \;+\; \int_A \rho H\,\mathbf{u}\cdot\mathbf{n}\,dA \;=\; 0\;,$$

where H is the Bernoulli quantity, $H = \frac{1}{2}|\mathbf{u}|^2 + \dfrac{p}{\rho} + \Phi$. (This is sometimes interpreted as saying that H is the 'transportable energy', or the 'advected energy', per unit mass.)

10. How high can water rise up one's arm hanging in the river from a lazy ($1\,\mathrm{m\,s^{-1}}$) punt? [Use Bernoulli on surface streamline, where $p = 1$ atmosphere.]

11. A flat-bottomed barge closely fits a canal, so that while it travels very slowly it still generates a fast $5\,\mathrm{m\,s^{-1}}$ current under it. Estimate how much lower in the water the barge lies as a result of this current. [*Hint:* Archimedes when stationary. Flow reduces pressure, so have to go deeper to get same pressure on long bottom.]

12. Waste water flows into a large tank at $10^{-4}\,\mathrm{m^3\,s^{-1}}$ and out of a short exit pipe of cross-section $4\times 10^{-5}\,\mathrm{m}^2$ into the air. The flow has reached a steady state. Estimate the height of the free surface of the water in the tank, relative to the height of the exit pipe.

13. A water clock is an axisymmetric vessel with a small exit pipe in the bottom. Find the shape of vessel for which the water level falls equal heights in equal intervals of time.

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