Mathematical Tripos Part IB
FLUID DYNAMICS

E.J. Hinch
November 2004

Example Sheet 2

1. Show that $\mathbf{u}(\mathbf{x}+\delta\mathbf{x}) - \mathbf{u}(\mathbf{x})$ can be written as $(e_{ij} + \frac{1}{2}\varepsilon_{jik}\omega_k)\delta x_j + O(|\delta\mathbf{x}|^2)$, where $e_{ij} = e_{ji}$ and $\omega_k = (\nabla \times \mathbf{u})_k$. Find e_{ij} and ω_k for the case of linear shear flow $\mathbf{u} = (y, 0, 0)$, and for this case sketch the streamlines of the two flows $e_{ij}x_j$ and $\frac{1}{2}\varepsilon_{jik}\omega_k x_j$.

2. Calculate the vorticity $\boldsymbol{\omega}$ of the velocity field

$$u = -\alpha x - yrf(t), \quad v = -\alpha y + xrf(t), \quad w = 2\alpha z$$

where $r^2 = x^2 + y^2$. Show that $\nabla.\mathbf{u} = 0$ for any function $f(t)$, and that the vorticity equation is satisfied if and only if $f(t) \propto \mathrm{e}^{3\alpha t}$.

Calculate the velocity components u_r and u_θ in cylindrical polar coordinates. Consider a material curve that starts at $t = 0$ as $z = 0$, $r = a_0$. Show that this becomes $z = 0$, $r = a_0 e^{-\alpha t}$. Verify the circulation theorem for this material curve.

3. If $\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{x}$ (uniform rotation), show that the vorticity is $\boldsymbol{\omega} = 2\boldsymbol{\Omega}$. For a two-dimensional flow $\big(u(x,y), v(x,y), 0\big)$, show that $\boldsymbol{\omega} = (0, 0, -\nabla^2\psi)$, where ψ is the streamfunction.

A long cylinder filled with water has elliptical cross-section with major and minor semi-axes a and b. While $t < 0$, both the cylinder and the water within it rotate about the axis of the cylinder with uniform angular velocity $(0, 0, \Omega)$. Sketch the streamlines, noting that they intersect the elliptical boundary of the cylinder. (Why?)

At $t = 0$, the cylinder is suddenly brought to rest. Assuming that the flow remains two-dimensional, what does the vorticity equation say about $\nabla^2\psi$ for $t > 0$? Verify that the flow for $t > 0$ can be described by

$$\psi = \frac{a^2b^2\Omega}{a^2+b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

in suitable coordinates. Sketch the streamlines.

4. A two-dimensional irrotational flow occupies the half-space $y < 0$ and is given by the velocity potential $\phi = \mathrm{e}^{ky}\sin kx \quad (k > 0)$. Show that the flow is incompressible. Calculate the velocity field $\mathbf{u}$ and the stream function $\psi(x, y)$. Sketch the streamlines.

5. A sphere of radius a moves with constant velocity U in a fluid otherwise at rest. How far ahead of the sphere is there a disturbance of magnitude $\frac{1}{20}U$ relative to the far-field? Show that the acceleration of a fluid particle at distance x ahead of the centre of the sphere is

$$3U^2\left(\frac{a^3}{x^4} - \frac{a^6}{x^7}\right) .$$

6. Write down the velocity potential $\phi(r, \theta, z)$ for the axisymmetric flow produced by a point source of strength m located at the origin, in the presence of a uniform stream $(0, 0, U)$. Show that there is a stagnation point at $(0, 0, -\frac{1}{2}a)$, where $a = (m/\pi U)^{1/2}$. Sketch the streamlines. Show that the Stokes streamfunction is given by $\Psi = \frac{1}{2}Ur^2 - mz/4\pi(z^2 + r^2)^{1/2}$. From the sketch and the streamfunction show that ϕ represents the flow around a semi-infinite body whose radius tends to a far downstream.

Sketch the flow if the source is re-located at $(0, 0, -l)$ and a sink of equal strength introduced at $(0, 0, l)$.

7. An orifice in the side of a large open vessel full of water leads smoothly into a horizontal tube of uniform cross-section and length L. The diameter of the tube is small in comparison with L and with the size of the vessel and the depth h of the orifice below the free surface. A plug at the end of the tube is suddenly removed and the water begins to flow. Neglecting small changes in h, show from the relevant form of Bernoulli's theorem, or otherwise, that the outflow velocity at subsequent times t is approximately

$$\sqrt{2gh}\tanh\left(\frac{t\sqrt{2gh}}{2L}\right) .$$

Deduce that the time it takes for the flow to accelerate to a fraction $(e^2-1)/(e^2+1) = 0.7616$ of its limiting value is $2L/\sqrt{(2gh)}$, and verify that this time is about 3 seconds for a garden hose $L = 9$ m long supplied by a rainwater tank with $h = 1.8$ m.

8. A U-tube consists of two long uniform vertical tubes of different cross-sectional areas A_1, A_2 connected at the base by a short tube of large cross-section, and contains an inviscid, incompressible fluid whose surface, in equilibrium, is at height h above the base. Derive the equation governing the nonlinear oscillations of the displacement ζ of the surface in the tube of cross-section A_2

$$(h+r\zeta)\frac{d^2\zeta}{dt^2} \;+\; \frac{r}{2}\left(\frac{d\zeta}{dt}\right)^2 \;+\; g\zeta \;=\; 0 \quad \text{where} \quad r = 1 - A_2/A_1 .$$

[*Hint:* Take $\phi = 0$ at the bottom of the U-tube, and remember that the irrotational form of Bernoulli's theorem involves $\partial\phi/\partial t$, implying time differentiation at a fixed $\mathbf{x}$.]

9. A sphere is immersed in an infinite calm ocean of incompressible fluid of density ρ. Its radius is given by $R(t) = a + b\sin nt$, where a, b and n are constants with $b < a$. The fluid moves radially under no external forces and the constant pressure at infinity is P. If $a \geq 5b$, show that the maximum pressure attained on the surface of the sphere is $P + \rho n^2 b(a-b)$. What is the corresponding formula if $b < a \leq 5b$?

10. A rigid disk of radius R is at a height $h(t)$ above a fixed plane $z = 0$, with fluid filling the gap between them, and $h \ll R$. Neglecting end effects from near the edge of the disk, show that the flow in the gap is described by

$$\phi \;=\; \frac{\dot{h}}{2h}\left(z^2 - \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2\right) ,$$

where the dot denotes the time derivative. Assuming that the pressure at the edge of the disk is constant, find the pressure distribution under the disk and hence the force on the fixed plane. Explain how the pressure distribution accelerates the radial flow in the cases $\dot{h} > 0$ and $\dot{h} < 0$.

11. A rigid sphere of radius a executes small amplitude oscillations (with a velocity $\mathbf{U}(t)$) while immersed in a fluid contained within a larger concentric fixed sphere of radius b. Find the velocity potential for the induced fluid motion. Neglecting terms quadratic in the amplitude, show that the pressure distribution over the surface of the moving inner sphere is

$$\rho\dot{U}a\cos\theta(a^3 + \tfrac{1}{2}b^3)/(b^3 - a^3)$$

(where θ is the angle with $\mathbf{U}$) and hence find the force exerted by the fluid on it? Why is the force on the outer fixed sphere different? Comment on the case of a tight fit.

Please email corrections/comments to E.J.Hinch@damtp.cam.ac.uk